Mathematics in Digital Images

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A Grayscale Image

A digital 8-bit grayscale image of $M \times N$ pixels can be thought of mathematically as a rectangular array of MN numbers between 0 and 255.

A Grayscale Image: Closer to the Bear Eye

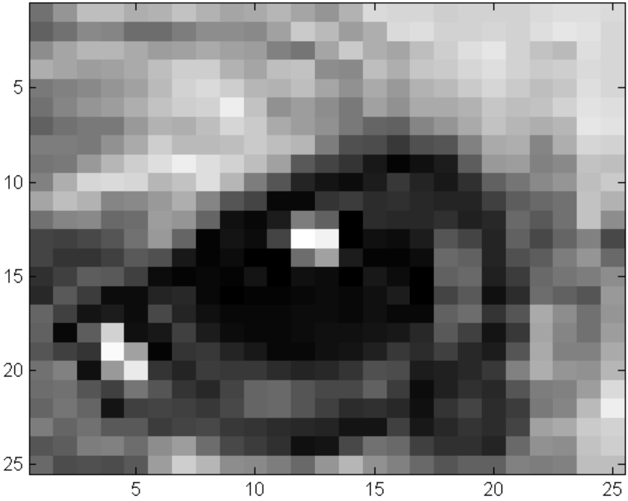
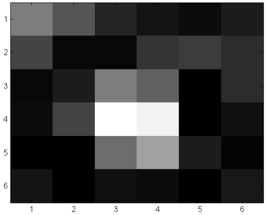


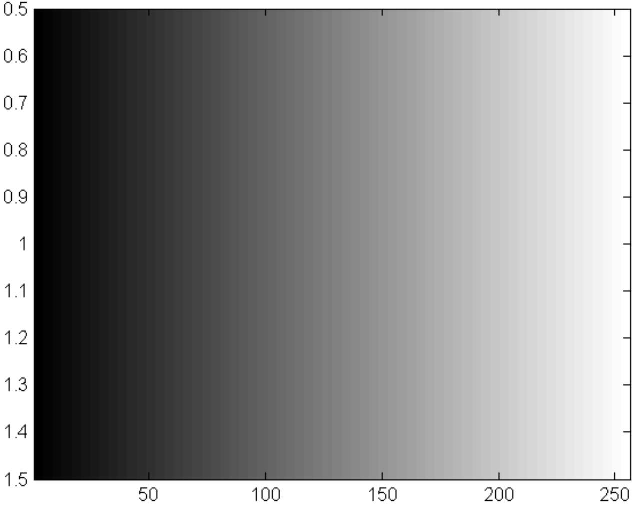
Image representation as a Matrix



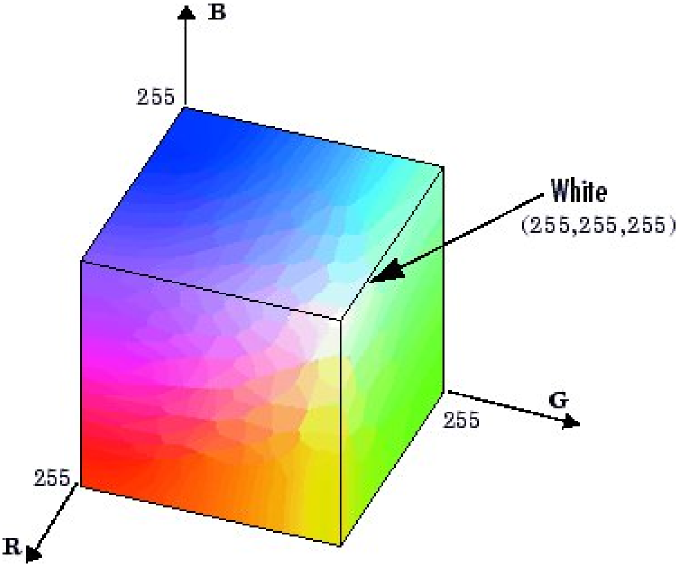
$$\begin{bmatrix} 126 & 87 & 39 & 22 & 13 & 32 \\ 68 & 9 & 10 & 54 & 61 & 47 \\ 10 & 31 & 125 & 98 & 3 & 47 \\ 15 & 70 & 253 & 240 & 4 & 19 \\ 2 & 2 & 111 & 161 & 31 & 5 \\ 24 & 4 & 20 & 15 & 1 & 25 \end{bmatrix}$$

The Grayscale Colormap

$$2^8 = 256$$

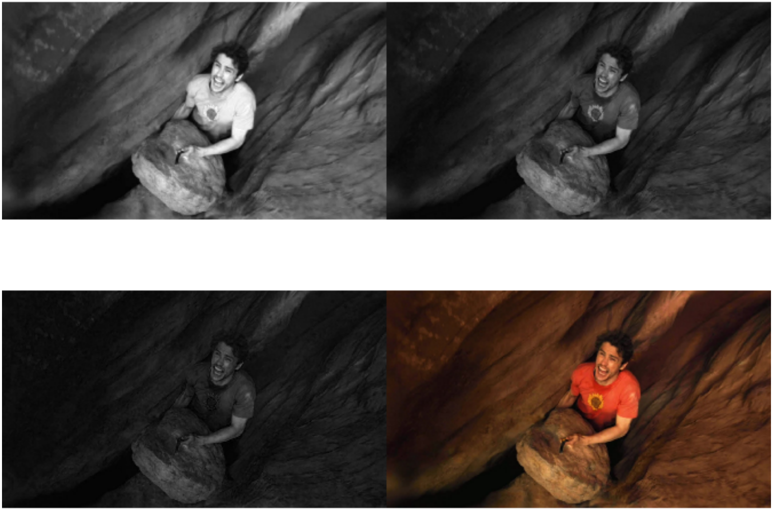


The RGB Colorcube

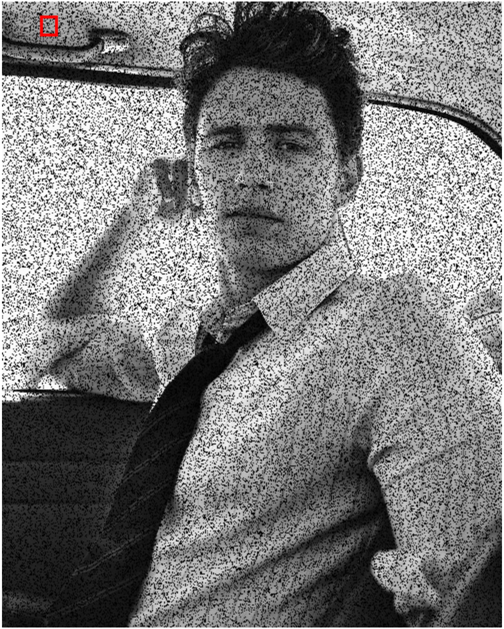


Color Images

Color Images



Correcting Images: Noise



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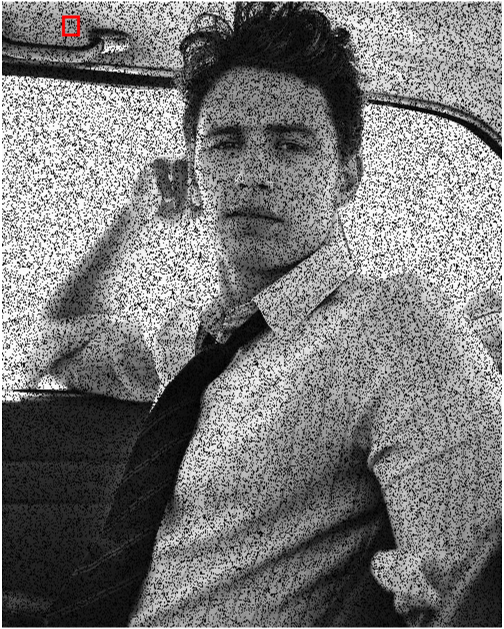
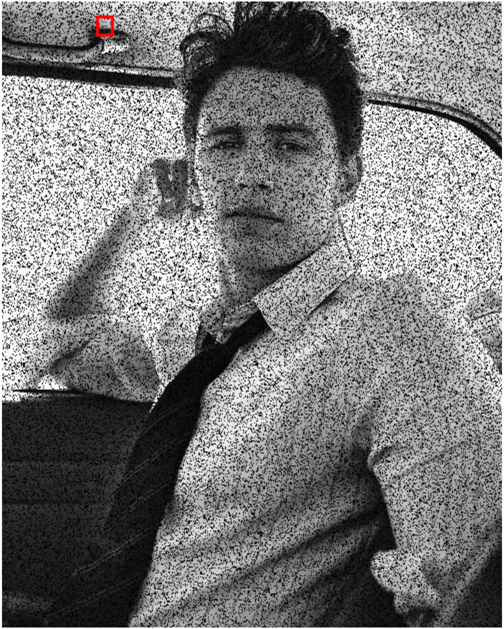
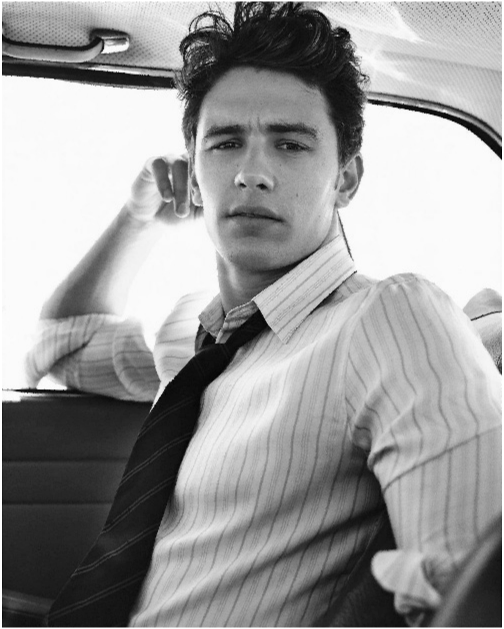


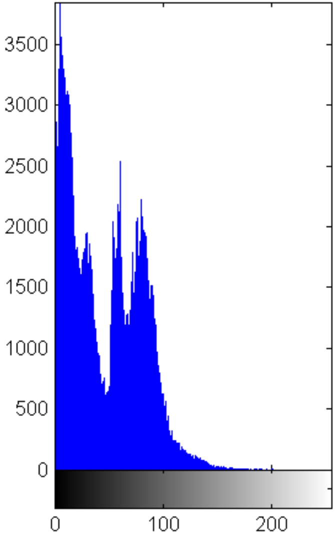


Correcting Images: Noise

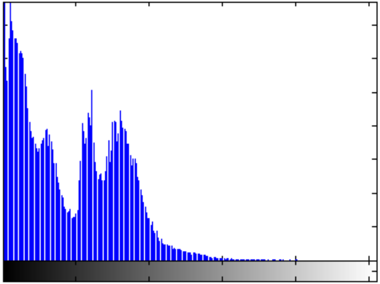


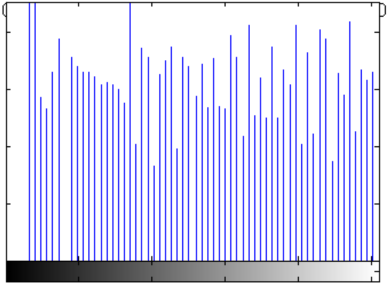
Correcting Images: Contrast

Correcting Images: Histograms

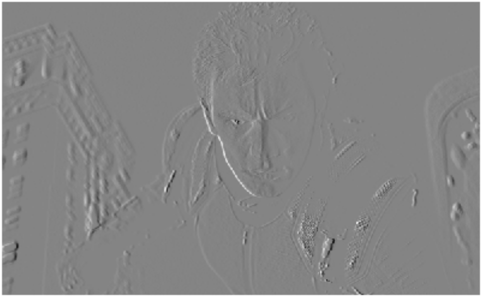


Correcting Images: Histogram Equalization





Edges and the Gradient



Functions have many representations. The Taylor series of f about point a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

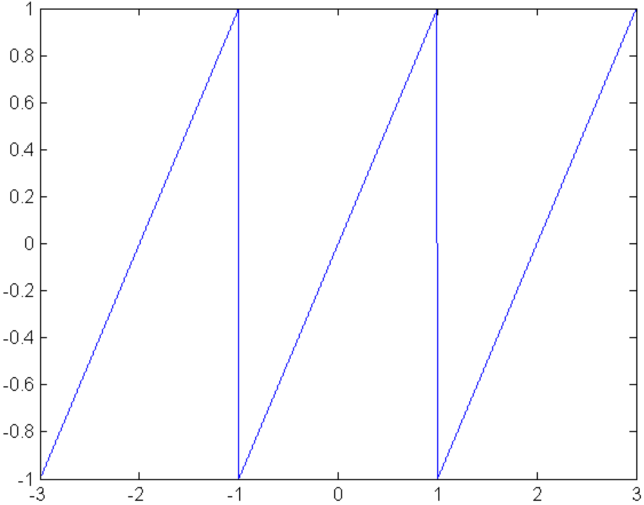
Functions have many representations. The Taylor series of f about point a is

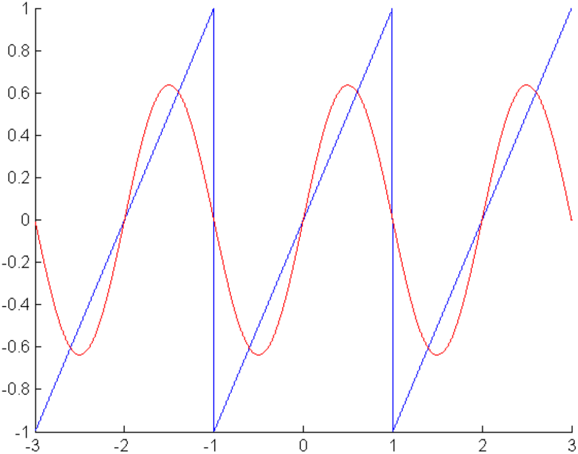
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We can also express f as the sum of sine waves. This is called the Fourier Series of f

$$f(x) = \sum_{n=0}^{\infty} a_n sin(nx) + b_n cos(nx)$$

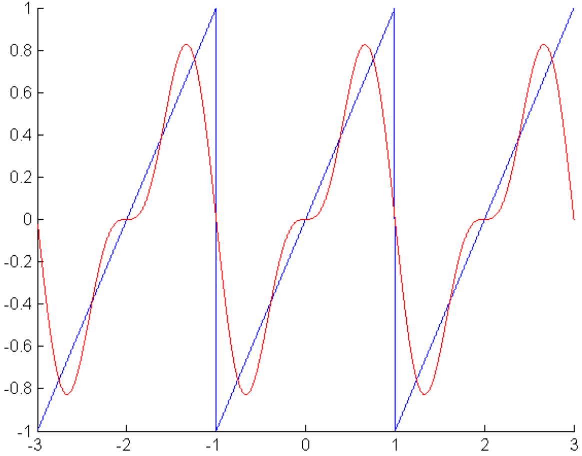
where a_n and b_n are the Fourier coefficients.


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3

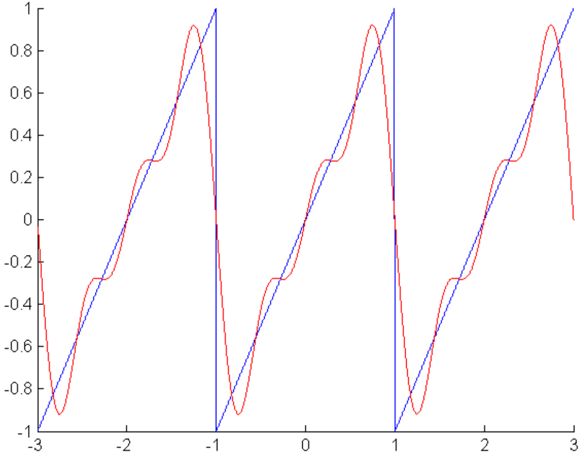


$$g(x) = \frac{2}{\pi}\sin(\pi x)$$

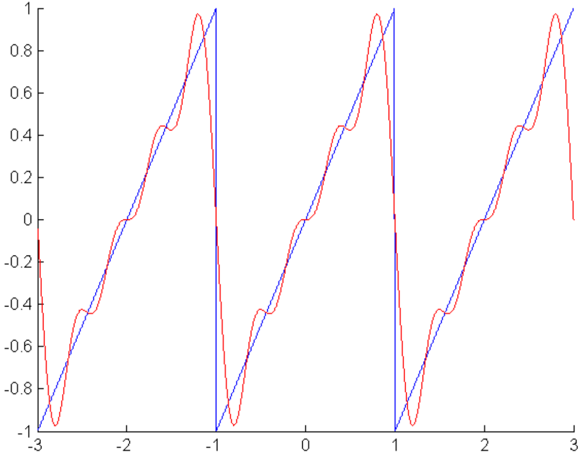
The Fourier Series



$$g(x) = \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x)$$

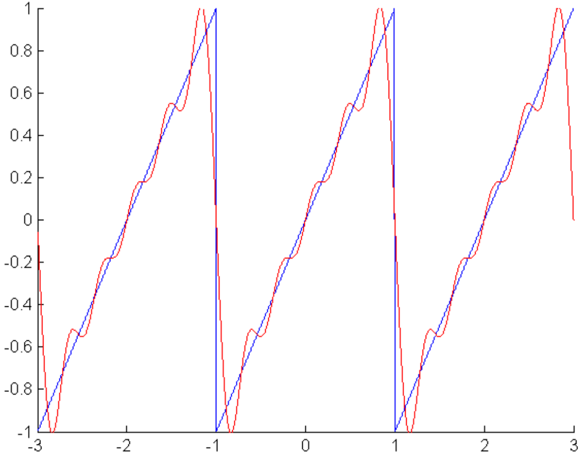


$$g(x) = \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x)$$



$$g(x) = \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) - \frac{1}{2\pi}\sin(4\pi x)$$

The Fourier Series



$$g(x) = \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) - \frac{1}{2\pi}\sin(4\pi x) + \frac{2}{5\pi}\sin(5\pi x)$$

The Fourier coefficients can be obtained from $\hat{f}$ the *Fourier transform* of f which gives the image in *frequency space*:

$$\hat{f}(m,n) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\left(\cos 2\pi(\frac{mx}{M}+\frac{ny}{N}) - i\sin 2\pi(\frac{mx}{M}+\frac{ny}{N})\right)$$

The Fourier Transform of an Image

The Fourier coefficients can be obtained from $\hat{f}$ the *Fourier transform* of f which gives the image in *frequency space*:

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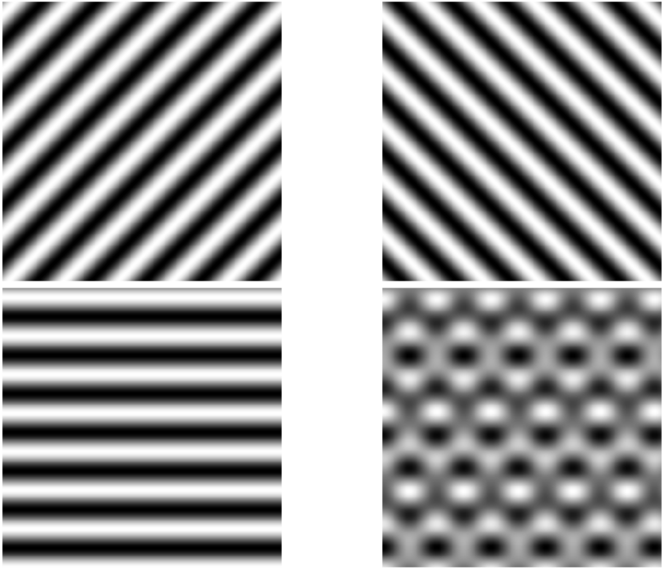
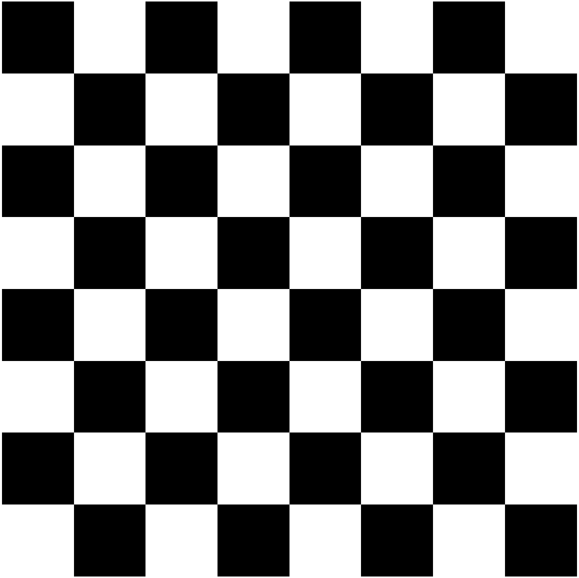
This transformation is reversible:

$$f(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f}(m,n)\left(\cos 2\pi(\frac{mx}{M}+\frac{ny}{N}) + i\sin 2\pi(\frac{mx}{M}+\frac{ny}{N})\right)$$

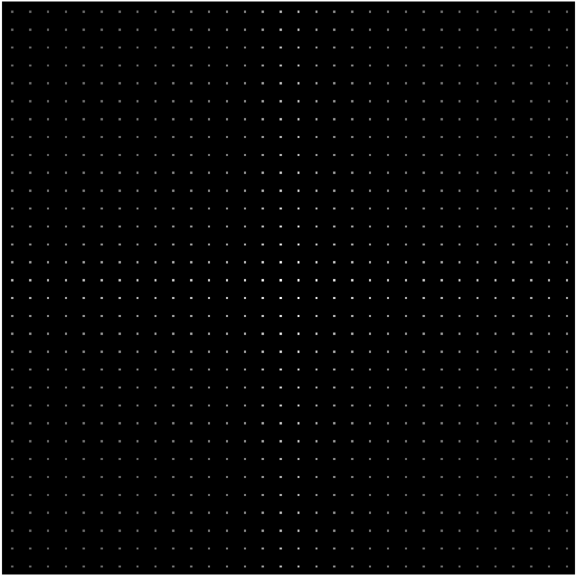
The Fourier Transform: Sine Waves



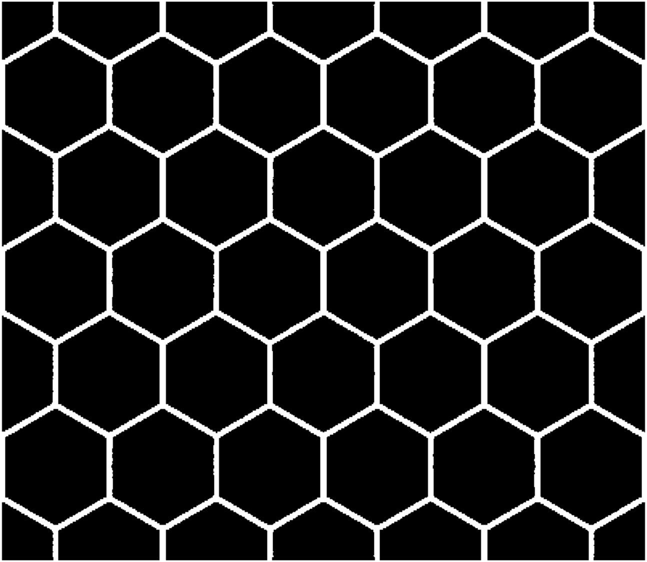
The Fourier Transform: Spatial Representation of an Image



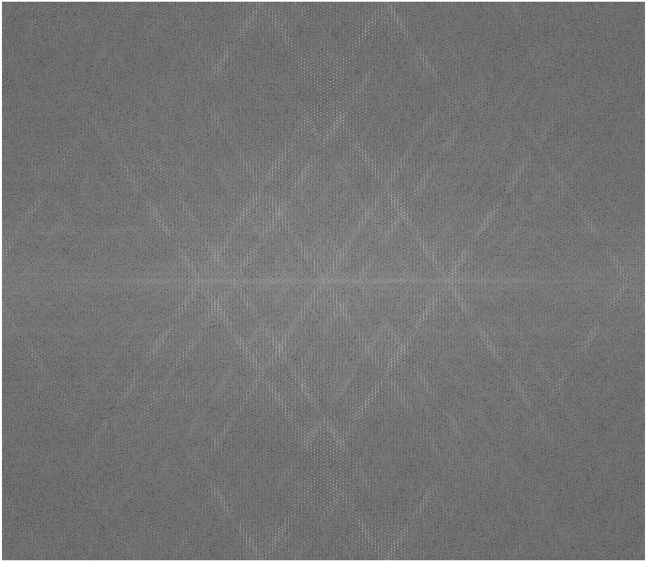
The Fourier Transform: Frequency Space



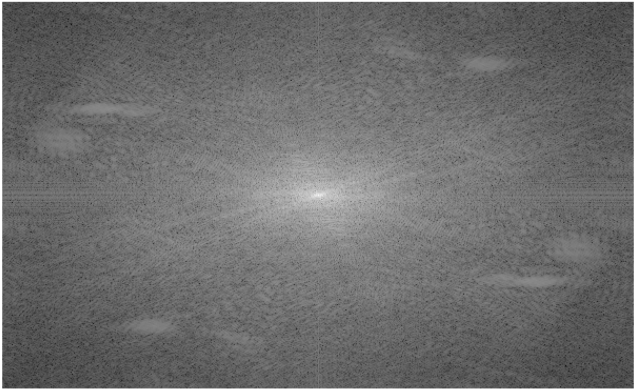
The Fourier Transform: Spatial Representation of an Image



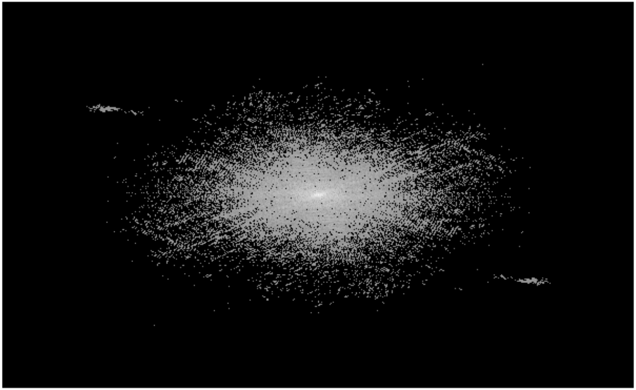
The Fourier Transform: Frequency Space



The Fourier Transform: Harry Osborn in Frequency Space



The Fourier Transform: Harry Osborn in Frequency Space



The Fourier Transform: 90% Compression Harry Osborn

Correcting Periodic Noise

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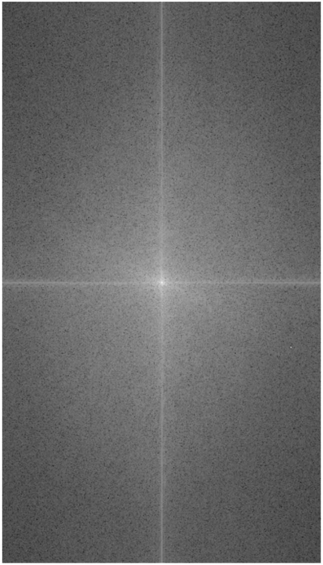

Correcting Periodic Noise