Math 1551-G
Fall 2015
Exam 3
30 October 2015
Time Limit: 50 Minutes

Name: Sols

This exam contains 8 pages (including this cover page) and 6 questions. There are 47 points in total. Write explanations clearly and in complete thoughts. No calculators or notes may be used. Put your name on every page. You must **include units** on quantities that carry units.

Grade Table

Question	Points	Score
1	9	
2	12	
3	8	
4	8	
5	5	
6	5	
Total:	47	

Formal Symbols Crib Sheet

$f : A \to B$	function with domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\lim_{x\to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x		

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f \circ g$	$(f' \circ g)g'$
f^{-1}	$\frac{1}{f' \circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x \rvert$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\mathrm{arccsc} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\mathrm{arcsec} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\mathrm{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

Geometry Crib Sheet

Pythagorean Identity $a^2+b^2=c^2$

Circle: radius r	$A=\pi r^2$	$c=2\pi r$
Box: dimensions x, y, z	$V=xyz$	$A=2(yz+xz+xy)$
Sphere: radius r	$V=\frac{4}{3}\pi r^3$	$A=4\pi r^2$
Right pyramid: height h dim x, y	$V=\frac{1}{3}hxy$	$A=xy+x\sqrt{(y/2)^2+h^2}+y\sqrt{(x/2)^2+h^2}$
Cylinder: height h radius r	$V=\pi hr^2$	$A=2\pi r(h+r)$
Right Cone: height h radius r	$V=\frac{\pi}{3}hr^2$	$A=\pi r\left(r+\sqrt{r^2+h^2}\right)$
Torus: radii $R>r$	$V=2\pi^2 r^2 R$	$A=4\pi^2 rR$
Tetrahedron: edge x	$V=\frac{1}{6\sqrt{2}}x^3$	$A=\sqrt{3}x^2$
Octahedron: edge x	$V=\frac{\sqrt{2}}{3}x^3$	$A=2\sqrt{3}x^2$
Dodecahedron: edge x	$V=\frac{15+7\sqrt{5}}{4}x^3$	$A=3\sqrt{20+10\sqrt{5}}x^2$
Icosahedron: edge x	$V=\frac{5(3+\sqrt{5})}{12}x^3$	$A=5\sqrt{3}x^2$

1. (a) (3 points) Let f be the real valued function with $f(x) = \sin(x^2)$. Compute the differential df.

$$df = 2x\cos(x^2)\,dx$$

(b) (6 points) Let f be a differentiable real function. Consider the graph $y = f(x)$ of the function and the tangent line at the point $(a, f(a))$. Which of the following statements are true of the tangent line? Circle all that apply.

A. the slope of the tangent line is $f'(a)$

B. the tangent line has equation $y - a = f'(a)(x - f(a))$

C. the tangent line has equation $y = f(a) + f'(x)(x - a)$

D. the tangent line is the best linear approximation to f near a

E. the tangent line intersects the graph in exactly one point

The correct one is

$$y = f(a) + f'(a)(x-a)$$

False because for example:

2. The table below contains some values of real functions f and g and their derivatives f' and g' evaluated at different values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	2	3	4
1	5	6	7	8
2	9	10	11	12
3	13	14	15	16

(a) (3 points) What is the derivative of the quotient f/g at 0?

$(f/g)'(0) = \dfrac{f'(0)\cdot g(0) - f(0)\,g'(0)}{g(0)^2} = \dfrac{2\cdot 3 - 1\cdot 4}{3^2} = \dfrac{2}{9}$

(b) (3 points) What is the derivative of the inverse function of f at 13?

$(f^{-1})'(13) = \dfrac{1}{f'(f^{-1}(13))} = \dfrac{1}{f'(3)} = \dfrac{1}{14}$

(c) (3 points) What is the derivative of the composition $\arctan \circ f$ at 1?

$(\arctan \circ f)'(1) = \dfrac{1}{1+f(1)^2}\cdot f'(1) = \dfrac{6}{1+5^2} = \dfrac{3}{13}$

(d) (3 points) Let $h(x) = g^{-1}(3x)$. What is the derivative of h at 5?

$h'(5) = 3\cdot (g^{-1})'(3x)\Big|_{x=5} = 3\cdot\dfrac{1}{g'(g^{-1}(3x))\big|_{x=5}}$

$= 3\,\dfrac{1}{g'(g^{-1}(15))} = \dfrac{3}{g'(3)} = \dfrac{3}{16}$

3. (a) (3 points) Find an expression for $\frac{dy}{dx}$ in terms of y and x if

$$e^{y^2} = y + x$$

$$e^{y^2} \cdot 2yy' = y' + 1$$

$$\Rightarrow y' = \frac{1}{2ye^{y^2} - 1} = \frac{1}{2y(y+x) - 1}$$

(b) (5 points) Find an expression for the second derivative $\frac{ds^2}{dt^2}$ in terms of s and t if

$$\ln|s| = t^3$$

$$\frac{s'}{s} = 3t^2$$

$$s' = 3t^2 s$$

$$s'' = 3t^2 s' + 6ts$$

$$= 3t^2(3t^2 s) + 6ts$$

$$= (9t^4 + 6t)s$$

4. (a) (5 points) Use a first order approximation to find a rational approximation for $29^{\frac{1}{3}}$.

$$29^{\frac{1}{3}} = (27+2)^{\frac{1}{3}} \cong 27^{\frac{1}{3}} + \frac{1}{3} 27^{-2/3} \cdot 2$$

$$= 3 + \frac{2}{27}$$

Using $f(x+h) \cong f(x) + f'(x) \cdot h$

(b) (3 points) Estimate the percent error in your approximation.

$$\frac{df}{f} = \frac{2/27}{3} = \frac{2}{81} \sim 2.5\%$$

5. (5 points) A spherical balloon is inflating with helium at a rate of $144\pi \text{cm}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 3cm?

$$\frac{dV}{dt} = +144\pi \ \text{cm}^3/\text{min}.$$

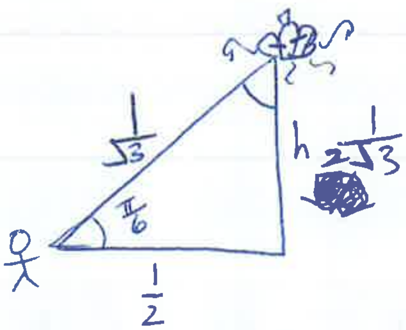
$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{144\pi \ \text{cm}^3/\text{min}}{4\pi \cdot 3^2 \ \text{cm}^2} = \frac{12^2}{12 \cdot 3} \ \text{cm}/\text{min}$$

$$= 4 \ \frac{\text{cm}}{\text{min}}$$

6. (5 points) Charlie Brown and Linus see the Great Pumpkin arise straight into the air from a pumpkin patch which they know is $\frac{1}{2}$ mile away. Charlie can estimate the height using the angle his line of sight to the pumpkin makes with the horizon. He measures the angle as $30°$ and its instantaneous rate of change as $12°$ per minute. How fast is the Great Pumpkin moving?



$$h = \frac{1}{2}\tan\theta \overset{\text{init}}{=} \frac{1}{2}\tan\left(\frac{\pi}{6}\right) mi = \frac{1}{2\sqrt{3}} \; mi$$

$$\frac{dh}{dt} = \frac{1}{2}\sec^2\theta \frac{d\theta}{dt}$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2 12 \cdot \frac{\pi}{180} \text{ mile/minute}$$

$$= \frac{24}{\sqrt{3}} \frac{\pi}{180} \text{ mi/min}$$

$$= \frac{8}{180} \pi\sqrt{3} \text{ mi/min}$$

$$= \frac{2}{45} \pi\sqrt{3} \text{ mi/min}$$

BONUS (5 points): Linus points out that Charlie's angle measurement has about $\pm 2°$ uncertainty and estimates that the Great Pumpkin rose to that height in 1 minute $\pm$ 20 seconds. What is the average velocity of the Great Pumpkin and what is the uncertainty in the calculation? Does the Great Pumpkin appear to be accelerating?

$$V = \frac{h}{t} = \frac{1}{2\sqrt{3}} \text{ mi/min}$$

$$h = \frac{1}{2}\tan\theta$$

$$dh = \frac{1}{2}\sec^2\theta \, d\theta$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2 \cdot 2° \frac{\pi}{180}$$

$$= \pm\frac{1}{45}\frac{\pi}{\sqrt{3}} \text{ mi}$$

$$dt = \pm\frac{1}{3} \text{ min}$$

$$dV = \frac{dh \cdot t - h\,dt}{t^2}$$

$$= \frac{\pm\frac{1}{45}\frac{\pi}{\sqrt{3}} \pm \frac{2}{45}\pi\sqrt{3}\,\frac{1}{3}}{1^2} \text{ mi/min}$$

$$= \frac{\pi}{45}\sqrt{3} \text{ mi/min}$$

That's 50% uncertainty!

The Great Pumpkin is accelerating since the instantaneous velocity is greater than the average.