Section 3.8 : Derivatives of Inverse Functions and Logarithms

Chapter 3 : Differentiation

Math 1551, Differential Calculus

"Language serves not only to express thought but to make possible thoughts which could not exist without it." - Bertrand Russell

For example: logarithms express an inverse operation to exponentiation, and they can also make it possible to differentiate complicated functions more easily.

Section 3.8 Derivatives of Inverse Functions and Logarithms

Topics

1. Derivatives of logarithmic and exponential functions.
2. Logarithmic differentiation.

Learning Objectives

For the topics in this section, students are expected to be able to:

1. Apply logarithmic differentiation to differentiate functions.
2. Differentiate logarithmic and exponential functions.

Note that the textbook also explores the limit definition of e and the derivative rule for inverses, which we won't have time to explore and students are not expected to know.

Motivation

Problem 1

- We have rules for differentiating functions of the form $(f(x))^n$, where n is an integer. For example:

$$\frac{d}{dx}\left((2+\cos x)^4\right) =$$

- We don't yet have rules for differentiating:

$$(f(x))^{g(x)}$$

Problem 2

- The derivative rules we currently have can get tedious for complicated functions.

In this section we introduce methods that solve both problems.

Logarithms Review

Recall: $\log_e(x) = \ln(x)$, and $e = 2.71828\ldots$

Which of the following statements are correct?

a) $\ln x + \ln y = \ln(x + y)$

b) $\ln x \ln y = \ln(xy)$

c) $\ln x - \ln y = \dfrac{\ln x}{\ln y}$

d) $x = e^{\ln x}$ for all $x \in \mathbb{R}$

e) All of the above.

f) None of the above.

Natural Logarithms

Recall:

$$\ln(xy) =$$

$$\ln\left(\frac{x}{y}\right) =$$

$$e^{\ln x} =$$

$$\ln x^y =$$

$$\log_a x =$$

Example 1

Example: Differentiate $y = \ln x$ for $x > 0$.

Example 2

Example: Differentiate $y = \log_2 x$ for $x > 0$.

Example 3

Differentiate $y(t) = 5^{2t}$ for $t > 0$.

Example 4

Differentiate $u(x) = x^{3x}$.

Example 5

Compute $y'(x)$ at $x = 0$.

$$y = \frac{(x+2)^8 \sqrt{x^2+25}}{e^{-2x}(x^2+5)}$$

Summary

- If $u = u(x) > 0$, $a > 0$, $a \neq 1$, then:

$$\frac{d}{dx}\ln u = \frac{u'}{u}\ , \qquad \frac{d}{dx}a^u = a^u \ln(a)u'\ , \qquad \frac{d}{dx}\log_a u = \frac{u'}{u\ln a}$$

- To differentiate $y = (f(x))^{g(x)}$, use **logarithmic differentiation**.
  1) Take natural log: $\ln y =$
  2) Differentiate: $\frac{d}{dx}\ln y =$
  3) Solve for y'.
- Complicated functions can sometimes be more easily differentiated using logarithmic differentiation (as we saw in the last example).

Additional Examples (as time permits)

Differentiate the following functions.

a) $y = x^{\ln x}$

b) $y = x^{1-x}$