Section 3.11 : Linearization and Differentials

Chapter 3 : Differentiation

Math 1551, Differential Calculus

"In fact, everything we know is only some kind of approximation, because we know that we do not know all the laws as yet. Therefore, things must be learned only to be unlearned again or, more likely, to be corrected."
- Richard Feynman

In this lecture we investigate the **approximation** of functions using tangent lines and derivatives.

Section 3.11 Linearization and Differentials

Topics

1. Linear approximations
2. Differentials

Learning Objectives

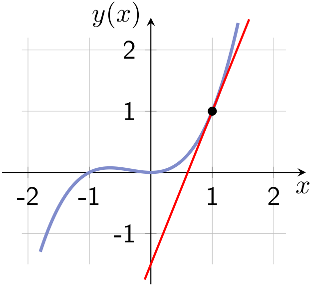
For the topics in this section, students are expected to be able to:

1. Construct differentials, and linearizations centered on a point.
2. Use differentials and linearizations to approximate function values, and to characterize how functions are changing near a given point.
3. Characterize the error made in a linear approximation.

Note that the textbook also explores sensitivity to change, which we won't have time to cover and students are not expected to know.

Motivation

$y(x) = \frac{1}{2}(x^3 + x^2)$ and its tangent line at $x = 1$ are shown below.



How well does the tangent line approximate the curve at $x = 0$? At $x = 0.5$? How do we quantify the error of the approximation?

First let's plot this curve in Desmos, and zoom in and out to get a better idea of what is going on.

Linearization

Definitions

If $f(x)$ is differentiable at $x = a$, then

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

is the **linear approximation** of f that is **centered** at a.

Example 1

Suppose $y = \sqrt[3]{x-1}$.

a) Construct a linearization of y centered at $x = 2$.

b) Use your linearization to approximate the value of $y(3)$.

Differentials

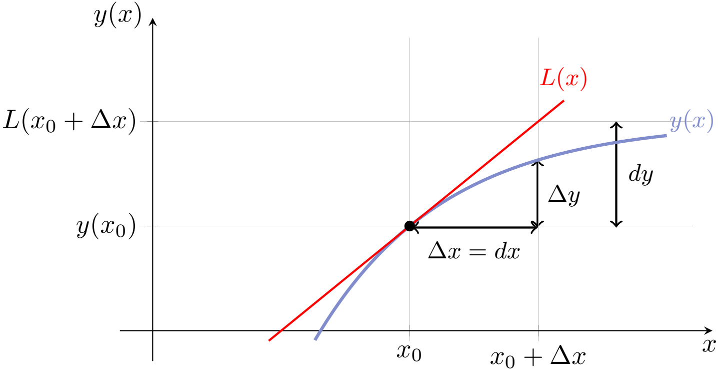
Definitions

Suppose $y = y(x)$ is a differentiable function. The **differential** dx is an independent variable, and the dependent variable dy is

$$dy = y'(x)\,dx$$

Often we set $dx = \Delta x$, which represents a change in x.

A Geometric Interpretation of Differentials



$L(x) =$ tangent line

$\Delta x = dx =$ change in x

$\Delta y =$ change in height of $y(x)$

$dy =$ change in height of tangent line

Example 2

The radius of a sphere, r, is 10 cm. If the radius increases by 0.01 cm, use a linear approximation centered at $r = 10$ to approximate the change in volume.

Approximation Error

As we move from $x = x_0$ to $x = x_0 + \Delta x$, we can describe the error in the approximation of $y = y(x)$ with $L(x)$, using the following.

Definition

$$
\begin{aligned}
\text{approximation error} &= |L(x_0 + \Delta x) - y(x_0 + \Delta x)| \\
&= |(L(x_0 + \Delta x) - L(x_0)) - (y(x_0 + \Delta x) - y(x_0))| \\
&= |dy - \Delta y|
\end{aligned}
$$

Additional Examples (as time permits)

1. Approximate the value of $\sqrt{4.04}$ using a linearization. Then construct an equation that can be used to compute the approximation error.
2. The radius of a circular disk, r, was measured to be 2 cm, with a measurement error of 0.1 cm.
   a) Construct a linearization centered at $r = 2$ that can be used to estimate the area of the disk.
   b) Use your linearization to estimate how large the area could be.
3. Suppose $f(x) = x^3 - 2$, $x_0 = 2$, $dx = 0.1$.
   a) Construct the linearization of f at $x = x_0$.
   b) Calculate $\Delta f = f(x_0 + dx) - f(x_0)$.
   c) Calculate the differential df at x_0.
   d) Calculate the approximation error at x_0.
   e) Sketch f, its linearization at x_0, df, and Δf.