Section 4.2 : The Mean Value Theorem

Chapter 4 : Applications of Derivatives

Math 1551, Differential Calculus

Section 4.2 The Mean Value Theorem

Topics

1. Rolle's Theorem
2. The Mean Value Theorem (MVT)
3. Consequences of the MVT: theorems

Learning Objectives

For the topics in this section, students are expected to be able to:

1. Determine whether Rolle's Theorem and the Mean Value Theorem can be applied to a given function and interval.
2. Apply Rolle's theorem and the Mean Value Theorem to characterize the roots, or the rate of change of a function (for example, to identify where the derivative of a function is equal to a particular value).
3. Give examples of functions whose derivatives meet certain criteria by using the Mean Value Theorem.

Example

Determine how many roots $f(x) = x^3 + x - 2$ has, if any, on the interval $x \in [0, 3]$.

Rolle's Theorem

Theorem

If $f(x)$ is a continuous function defined on $[a, b]$, and is differentiable over (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Participation Activity: Index Card

- Please work in groups of two or three
- Each group submits **one** completed card
- Print full names at the top of your card
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work or for students working by themselves
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution
- Print today's date at the top, which is ____________

The activity consists of one or two of the examples in this lecture. Your instructor will pass out index cards.

Example

Your group is driving from Atlanta to Knoxville, TN. The trip requires about 210 miles of driving. Let $f(t)$ be the distance between where you started, as a function of time, t. Assume the speed limit is 70 mph for the entire journey.

a) If your group completes the trip after 2 hours of driving, sketch an example of what $f(t)$ could look like.

b) Is $f(t)$ a continuous function over a closed interval?

c) Is $f(t)$ a differentiable function?

d) Did you need to break the speed limit? Use your sketch and a derivative to justify your reasoning.

The Mean Value Theorem (MVT)

Theorem

If $f(x)$ is a continuous function defined on $[a, b]$, and is differentiable over (a, b). Then there is at at least one point, $c \in (a, b)$, where

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Example

Which functions satisfy the conditions of the MVT? For those that do, identify all values of c that satisfy $\frac{f(b)-f(a)}{b-a} = f'(c)$, on interval (a, b).

a) $f(x) = \sqrt{x+1}$, on $x \in [0, 3]$.

b) $f(x) = \dfrac{1}{x^2 + x}$, on $x \in [-2, 2]$.

Consequences of the MVT

Proofs for the following theorems are stated in the textbook. If time permits, we will prove all/most of these theorems in lecture.

1. If $f'(x) = 0$ on (a, b), then $f = C$ on (a, b), where $C \in \mathbb{R}$.
2. If $f'(x) = g'(x)$ on (a, b), then $f(x) = g(x) + C$ for all $x \in (a, b)$, where $C \in \mathbb{R}$.

Additional Example (if time permits)

1. Give a formula that could represent a function $f(x)$, that satisfies $f'(x) = g'(x)$ for $x \in (-1, 1)$, $g(x) = x^2 + 1$, and $f(0) = 2$.
2. Sketch a non-zero function $f(x)$ that satisfies $f'(x) = 0$ for all $x \in (-4, 2)$.
3. True or false: if f and g are differentiable functions and $f(x) - g(x) = 3$ for all $x \in (a, b)$, then $f'(x) = g'(x)$ for all $x \in (a, b)$.