Math 1551M2 **Name:** __________________________

Fall 2017

Exam 2

17 November 2017

Time Limit: 50 Minutes

This exam contains 7 pages (including this cover page) and 5 questions. There are 33 points in total. Write explanations clearly and in complete thoughts. No calculators or notes may be used. Put your name on every page.

Grade Table

Question	Points	Score
1	6	
2	6	
3	6	
4	9	
5	6	
Total:	33	

Formal Symbols Crib Sheet

$f : A \to B$	function with domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\lim_{x\to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x		

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f \circ g$	$(f' \circ g)g'$
f^{-1}	$\frac{1}{f' \circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x \rvert$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\text{arccsc} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\text{arcsec} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\text{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

Geometry Crib Sheet

Pythagorean Identity $a^2+b^2=c^2$

Circle: radius r	$A=\pi r^2$	$c=2\pi r$
Box: dimensions x, y, z	$V=xyz$	$A=2(yz+xz+xy)$
Sphere: radius r	$V=\frac{4}{3}\pi r^3$	$A=4\pi r^2$
Right pyramid: height h dim x,y	$V=\frac{1}{3}hxy$	$A=xy+x\sqrt{(y/2)^2+h^2}+y\sqrt{(x/2)^2+h^2}$
Cylinder: height h radius r	$V=\pi hr^2$	$A=2\pi r(h+r)$
Right Cone: height h radius r	$V=\frac{\pi}{3}hr^2$	$A=\pi r\left(r+\sqrt{r^2+h^2}\right)$
Torus: radii $R>r$	$V=2\pi^2r^2R$	$A=4\pi^2rR$
Tetrahedron: edge x	$V=\frac{1}{6\sqrt{2}}x^3$	$A=\sqrt{3}x^2$
Octahedron: edge x	$V=\frac{\sqrt{2}}{3}x^3$	$A=2\sqrt{3}x^2$
Dodecahedron: edge x	$V=\frac{15+7\sqrt{5}}{4}x^3$	$A=3\sqrt{20+10\sqrt{5}}x^2$
Icosahedron: edge x	$V=\frac{5\left(3+\sqrt{5}\right)}{12}x^3$	$A=5\sqrt{3}x^2$

1. (6 points) Suppose that f is a twice differentiable real function defined on the closed interval $[0, 10]$.

   (a) Suppose that $f(5) = 7$ and $f'(5) = 0$ and $f''(5) = -2$. Does f achieve a maximum at 5?

   A. Yes, $f(5)$ is the global maximum.

   B. Yes, $f(5)$ is a local maximum, but cannot be the global maximum.

   C. Yes, $f(5)$ is a local maximum, but we do not have enough information to know if it is the global maximum.

   D. No, $f(5)$ cannot be a local maximum.

   E. We do not have enough information to decide.

   Solution: C. The second derivative test guarantees that $f(5)$ is a local maximum, but we have no way of knowing what the function is like at other points.

   (b) In addition to the information above, suppose that $f(0) = f(1) = f(2) = 3$. What is the fewest possible number of critical points that f could have?

   Solution: 3. We have that 5 is a critical point and by the mean value theorem there is also a critical point in $(0, 1)$ and in $(1, 2)$.

   (c) In addition to the information above, suppose that $f'(2) = 21$. Give the linearization for f about 2.

   Solution: The linearization is $L(x) = 3 + 21(x - 3)$

2. (6 points) Consider the function $f(x) = x \ln |x|$ on its natural domain. Find any critical points, inflection points, and the intervals on which f is increasing, decreasing, concave up, and concave down.

The critical points are ________________.

f is increasing on ________________

f is decreasing on ________________

The inflection points are ________________.

f is concave up on ________________

f is concave down on ________________

Solution: Compute $f'(x) = \ln |x| + 1$. Then the derivative is 0 at $x = \pm\frac{1}{e}$ and does not exist at $x = 0$. So the critical points are $-\frac{1}{e}, 0, \frac{1}{e}$. We have that f' is positive, therefore increasing, on $(-\infty, -1/e) \cup (1/3, \infty)$. We have that f' is negative, therefore decreasing, on $(-1/e, 0) \cup (0, 1/e)$. Compute $f''(x) = 1/x$, which is never valued 0, but does not exist at $x = 0$. Considering the signs of f'', we see f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$ and $x = 0$ is an inflection point.

3. (6 points) A spherical bubble is made from 4π grams of fluid and inflating at a rate of π cm^3/sec. The thickness of the bubble is related to the density of the fluid

$$\rho = \frac{m}{A}$$

where m is the mass of the fluid and A is the surface area. How fast is the density ρ changing when the bubble volume is 8 cm^3?

$\frac{d\rho}{dt} =$ _________________________

Solution: The volume of a sphere is $V = \frac{4}{3}\pi r^3 = 8$ cm^3 and the surface area is $A = 4\pi r^2$. We are given that $\frac{dV}{dt} = 4\pi$ cm^3/sec, $V = 8$ cm^3, and that mass is constant at 4π grams. Since we are given V and $\frac{dV}{dt}$, solve for ρ in terms of V.

$$\rho = mA^{-1} = \frac{4\pi}{4\pi r^2}\text{g} = r^{-2}\text{g}$$

so

$$\frac{d\rho}{dt} = -2r^{-3}\frac{dr}{dt}\text{g}$$

relating $\pi cm^3/sec = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ we get $\frac{dr}{dt} = \frac{1}{4}r^{-2}$ cm^3/sec so by substituting $\frac{dr}{dt}$ and r

$$\frac{d\rho}{dt} = -\frac{1}{2}r^{-5}\text{ g cm}^3\text{/sec} = -\frac{1}{2}\left(\frac{\pi}{6}\right)^{5/3}\text{ g/cm}^2\text{sec}$$

4. (9 points) Let h be a function defined on the domain $[0, 5]$ with the rule

$$h(x) = |x - 1| + 2|x - 3|.$$

Compute the global maximum, minimum, and the arguments at which they occur. Give the intervals where h is increasing and decreasing.

The absolute maximum is ________________ which occurs at $x =$________________.

The absolute minimum is ________________ which occurs at $x =$________________.

h is increasing on ________________.

h is decreasing on ________________.

Solution: The first term of $h(x)$ has slope ± 1 and the second term ± 2, so the slope can never add up to 0. Since $|x - 1|$ is not differentiable at 1 and $|x - 3|$ is not differentiable at 3, we have the critical points must be 1 and 3. Then the extrema have to occur at $0, 1, 3$, or 5. We compute $h(0) = 7$, $h(1) = 4$, $h(3) = 2$, and $h(5) = 8$. So the max is 8 occuring at 5 and the min is 2 occuring at 3. Then h is decreasing on $(0, 1) \cup (1, 3)$ and increasing on $(3, 5)$.

5. (6 points) (a) Use the fact that $2^5 = 32$ and a linear approximation to compute a rational number approximating the irrational number $30^{2/5}$.

$30^{2/5} \approx$ ____________________

Solution: Linearize the function $f(x) = x^{2/5}$ about 32.

$$30^{2/5} \approx f(32) + f'(32) \cdot (30 - 32) = 32^{2/5} - 2 \cdot \frac{2}{5} \cdot 32^{-3/5} = 4 - \frac{1}{10} = 3.9$$

(b) You estimate the volume of a sphere to be $36\pi\text{cm}^3$ by submerging it in water and measuring the displacement, but the measurement has an uncertainty of $\pm\pi\text{cm}^3$. Compute the radius and find the uncertainty in the radius measurement.

Solution: Using $36\pi\text{cm}^3 = V = \frac{4}{3}\pi r^3$ so $r = 3$ cm. We have $\Delta V \approx \frac{dV}{dr}\Delta r$ so $\pm\pi = 4\pi r^2 \Delta r$ gives $\Delta r = \frac{1}{36}$ cm.