Things to Definitely Know

Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Pythagorean Identity

$$\cos^2\theta + \sin^2\theta = 1$$

Trigonometric Identities

$$\cos(u+v) = \cos u\cos v - \sin u \sin v$$
$$\sin(u+v) = \cos u \sin v + \sin u \cos v$$
$$\cos^2 u = \frac{1}{2}(1+\cos 2u)$$

I **First Order Differential Equations**

1. Linear Equation $y' + py = g$. Multiply by the integrating factor $\mu = e^{\int p}$:

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(t)g(t)\ dt + c\right)$$

   Compare with Variation of Parameters below. Examples: Tank Mixing, Continuously Compounded Interest, Velocity

2. Separable Equation: $y' = f(x)g(y)$. Seperate and integrate sides separately:

$$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx + c$$

   Solve for y when possible.

3. Exact Equation: $M(x,y)\ dx + N(x,y)\ dy = 0$. The equation is an exact differential form if

$$d\psi = \frac{\partial\psi}{\partial x}\ dx + \frac{\partial\psi}{\partial y}\ dy = M(x,y)\ dx + N(x,y)\ dy = 0$$

   Check exactness by checking that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If exact then the solutions are the level sets of a potential function $\psi(x,y) = k$.

$$\psi(x,y) = \int M\partial x + \int\left(N - \frac{\partial}{\partial y}\int M\partial x\right)\partial y$$

   Examples: Population Dynamics, Black Hole Evaporation, Newton's Law of Cooling

II **Euler's Method**

1 A solution to an initial value problem $y(t_0) = y_0$ and $y' = f(y,t)$ can be estimated numerically by a piecewise linear function.

2 *Euler's method* with step size h estimates a solution iteratively by setting

$$t_n = t_{n-1} + hy_n = y_{n-1} + hf(y_{n-1}, t_{n-1})$$

3 The Mean Value Theorem guarantees there is some unknown time $a \in (t_n, t_{n+1})$ such that

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(a)$$

4 The local truncation error in Euler's method can be bounded by

$$|y_n - y(t_n)| \leq \frac{h^2}{2}\max|y''|$$

5 The total local truncation error in Euler's method from t_0 to t_n can be estimated as

$$\approx \frac{h}{2}(t_n - t_0)\max|y''|$$

III **Nonlinear Autonomous Systems of Differential Equations:**

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \tag{1}$$

where x and y are functions of time t and f and g are functions of x and y only. More generally $\dot{\mathbf{x}} = F(\mathbf{x})$. Often not solveable analytically.

1 *Equilibrium points* or *critical points* or *stationary points* are points where the derivative $F(\mathbf{x})$ vanishes. In a 2 dimensional system, the equilibrium (x_0, y_0) satisfies

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}_{|x=x_0, y=y_0} = \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2 *Almost Linear Systems* A nonlinear equation may be approximated by a linear equation near equilibrium point $\mathbf{a}$ (or (x_0, y_0)) using the Taylor expansion $\frac{d}{dt}\mathbf{x} \approx F'(\mathbf{a})(\mathbf{x} - \mathbf{a})$ for $\mathbf{x}$ near $\mathbf{a}$, where F' is the Jacobian derivative. For a 2 dimensional system for (x, y) near (x_0, y_0)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{|x=x_0, y=y_0} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

3 *Stability Analysis* The Jacobian derivative at the equilibrium point gives stability:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

Equilibrium (x_0, y_0) is attracting if every eigenvalue of $J(x_0, y_0)$ is negative, repelling if every eigenvalue is positive, and a saddle if there are eigenvalues of mixed sign.

4 *Solution trajectories* are the curves in phase space (or x-y space) traced by solutions to equation (1). Trajectories are solutions to the (sometimes solveable) differential equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f(x,y)}{g(x,y)}$$

5 A differential equation or in general any dynamical system is *chaotic* if it is 1) sensitive to initial conditions, 2) the time evolution of any two regions eventually overlaps, and 3) every point is arbitrarily close to a periodic orbit.

6 Nonlinear differential equations might have *strange chaotic attractors* where solutions are chaotic. Strange attractors may be complicated sets, but nearby solutions will move toward the attractor.

7 Individual numerical solutions to chaotic equations are unreliable, but the locations and shapes of strange attractors can be estimated numerically.

IV **Homogenous Linear Systems:**

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{2}$$

where $\mathbf{A}$ is a $n \times n$ matrix valued function and $\mathbf{x}$ is a vector valued fucntion.

1 If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions to equation (2) then so is any linear combination $a\mathbf{x}(t) + b\mathbf{y}(t)$.

2 Equation (2) has n linearly independent solutions away from the discontinuities of $\mathbf{A}$.

3 Solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent in interval $[a, b]$ if and only if the Wronskian is non-zero in $[a, b]$

$$W(t) = \det[\mathbf{x}_1(t) \ldots \mathbf{x}_n(t)] \neq 0$$

4 If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent solutions then the matrix with $\mathbf{x}_1, \ldots, \mathbf{x}_n$ as column vectors is a *fundamental* matrix.

$$\chi(t) = [\mathbf{x}_1(t) \ldots \mathbf{x}_n(t)]$$

5 Subject to the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ with t_0 in interval $[a, b]$ with $\det \chi(t) \neq 0$ for every $t \in [a, b]$ then there is a unique solution

$$\mathbf{x}(t) = \chi(t)\chi^{-1}(t_0)\mathbf{x}_0$$

6 One can transform an n^{th} order linear differential eqation into a linear system:

$$y^{(n)} = a_0(t)y(t) + a_1(t)y' \ldots + a_{n-1}(t)y^{(n-1)}$$

becomes

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & & \\ \vdots & \mathbf{I}_{n-1} & \\ 0 & & \\ a_0(t) & \ldots & a_{n-1}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & & \ldots & 0 & 1 \\ a_0(t) & a_1(t) & \ldots & a_{n-2}(t) & a_{n-1}(t) \end{pmatrix}$$

V **Autonomous Homogenous Linear Systems:**

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{3}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a constant matrix and $\mathbf{x}$ is a vector valued function.

1 Eigenvalues λ are roots of the characteristic polynomial

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

Eigenvalues are real or come in complex conjugate pairs.

2 The *algebraic multiplicity* of the eigenvalue is its multiplicity as a root of the characteristic polynomial.

3 Eigenvectors are linearly independent solutions to

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

4 If $\mathbf{v}$ is an eigenvector associated to eigenvalue λ then

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a solution to Equation (3).

5 If $\mathbf{v}_\pm = \mathbf{u} \pm i\mathbf{w}$ are a pair of complex eigenvectors correpsponding to conjugate pair of eigenvalues $\lambda_\pm = \alpha \pm i\beta$ then the real part of the span of the solutions $e^{\lambda_+ t}\mathbf{v}_+$ and $e^{\lambda_- t}\mathbf{v}_-$ is the span of solutions $e^{\alpha}(\cos\beta t\mathbf{u} - \sin\beta t\mathbf{w})$ and $e^{\alpha}(\sin\beta t\mathbf{u} + \cos\beta t\mathbf{w})$

6 The rank of $(\mathbf{A}-\lambda\mathbf{I})$ is the number of linearly independent columns. $rank(\mathbf{A}-\lambda\mathbf{I})$ = number of linearly independent rows = number of pivots in the reduced row eschelon form = dimension of the range

7 The geometric multiplicity of λ is the number of linearly independent eigenvectors of $\mathbf{A}$ associated to λ. By the rank-nullity theorem, the geometric multiplicity of λ is $n - rank(\mathbf{A} - \lambda\mathbf{I})$

8 Algebraic multiplicity $\geq$ geometric multiplicity ≥ 1. If algebraic multiplicity of $\lambda >$ geometric multiplicity of λ, then λ is a *defective* eigenvalue and $\mathbf{A}$ is a *defective* matrix.

9 A *generalized eigenvector* is a vector $\mathbf{w}$ such that $(A - \lambda I)^k\mathbf{w} = 0$ for some k. The smallest such k is the rank of the generalized eigenvector $\mathbf{w}$.

10 For every $n \times n$ matrix $\mathbf{A}$ there is a basis of basis of generalized eigenvectors $V = [v_1|\ldots|v_n]$ that block-diagonalize $\mathbf{A}$ into *Jordan Canonical form.*

$$\mathbf{A} = VJV^{-1}$$

where J is a block diagonal matrix with *Jordan blocks* of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}$$

11 The columns of such a change of basis matrix V may be computed by forming *Jordan chains*

$$\mathbf{w}_n \xrightarrow{A-\lambda I} \mathbf{w}_{n-1} \xrightarrow{A-\lambda I} \ldots \xrightarrow{A-\lambda I} \mathbf{w}_1 \xrightarrow{A-\lambda I} 0$$

of generalized vectors by iteratively choosing solutions to $(A - \lambda I)\mathbf{w}_{k+1} = w_k$ for $k = 0 \ldots n$ where $w_0 = 0$. The lengths of all Jordan chains for eigenvalue λ sum to the algebraic multiplicity m. The number of Jordan chains for eigenvalue λ is the geometric multiplicity.

12 If $\mathbf{w}_n \overset{A-\lambda I}{\longrightarrow} \mathbf{w}_{n-1} \overset{A-\lambda I}{\longrightarrow} \ldots \overset{A-\lambda I}{\longrightarrow} \mathbf{w}_1 \overset{A-\lambda I}{\longrightarrow} 0$ is a Jordan chain then there is a linearly independent solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ for every generalized eigenvector in the chain given by

$$e^{\lambda t}\left(\mathbf{w}_k + t\mathbf{w}_{k-1} + \ldots + \frac{t^{k-1}}{(k-1)!}\mathbf{w}_1\right)$$

for each $k = 1 \ldots n$.

13 Equivalently, if λ has algebraic multiplicity m and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ span the nullspace of $(\mathbf{A} - \lambda\mathbf{I})^m$, then for $k = 1, \ldots, m$

$$\mathbf{x}_k(t) = \frac{t^{m-1}e^{\lambda t}}{m-1!}(\mathbf{A} - \lambda\mathbf{I})^{m-1}\mathbf{v}_k + \ldots + e^{\lambda t}\mathbf{v}_k$$

gives m linearly independent solutions to equation (3).

14 If $\chi(t)$ is a fundamental matrix then the *matrix exponential* is the unique fundemental matrix normalized at 0

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n = \chi(t)\chi^{-1}(0)$$

15 The general solution to equation (3) is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c}$ where $\mathbf{c} \in \mathbb{R}^n$ is a constant vector.

16 $\mathbf{A}$ is invertible if and only if it has only non-zero eigenvalues if and only if $\det \mathbf{A} \neq 0$.

17 If $\mathbf{A}$ is invertible than the dynamical system $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$ has exactly one equilibrium point: The origin. The equilibrium point is *stable* or *attracting* if all the eigenvalues of $\mathbf{A}$ have negative real part, *unstable* or *repelling* if all the eigenvalues have positive real part, and a *saddle* of *semistable* if the eigenvalues have mixed signs. If the eigenvalues are complex then solutions spiral or oscillate. If the eigenvalues are completely imaginary, then the equilibrium point is a spiral center.

18 The *phase portrait* of the dynamical system $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$ is a visual discription of solution trajectories in $\mathbb{R}^n$ and enable a graphical analysis of long term solution behavior ($t \to \infty$). Solution trajectories should demonstrate the eigenlines, the dominant solution behavior, and the direction in which all trajectories are followed. Direction fields are also a plus.

19 To find the inverse of $\mathbf{A}$

$$rref[\ \mathbf{A} \mid \mathbf{I}\ ] = [\ \mathbf{I} \mid \mathbf{A}^{-1}\ ]$$

20 If $\mathbf{A}$ is invertible then $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$ can be solved by a change of variables to $\mathbf{y} = \mathbf{x} + \mathbf{A}^{-1}\mathbf{b}$ and solving $\frac{d}{dt}\mathbf{y} = \mathbf{A}\mathbf{y}$.

VI Nonhomogenous Linear Systems:

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t) \tag{4}$$

1. Find a fundemental set of solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to the homogenous equation $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)$. Let $\chi = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ be the corresponding fundamental matrix.
2. A *particular solution* to the nonhomogenous equation is given by

$$\mathbf{x}_p(t) = \chi \int \chi^{-1}\mathbf{g}(t)\ dt$$

3. The general solution to the nonhomogenous equation (4) is then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t)$$

or

$$\mathbf{x}(t) = \chi\mathbf{c} + \mathbf{x}_p = \chi\left(\int \chi^{-1}g\ dt + \mathbf{c}\right)$$

VII Second Order Linear Equations:

$$y'' + p(t)y' + q(t)y = g(t) \tag{5}$$

1. Given two solutions $y_1(t)$ and $y_2(t)$ the Wronskian is

$$W(t) = \det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

The Wronskian is nonzero wherever the solutions are linearly indepedent.

2. The general solution to equation (5) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

where y_1 and y_2 are linearly indepedent solutions to the homogenous equation

$$y'' + p(t)y' + q(t)y = 0$$

and y_p is a particular solution to equation (5).

3. Variation of Parameters: If y_1 and y_2 are homogenous solutions to equation (5) with Wronskian W then a particular solution is

$$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\,dt$$

VIII **Constant Coefficient Second Order Linear Equations:**

$$my'' + by' + ky = g(t) \tag{6}$$

1. If $m, b, k > 0$ the equation can be interpreted as the Newtonian equation of motion for a mass spring system of mass m, damping constant b, and Hooke constant k under a driving force of $g(t)$.
2. The characteristic polynomial of equation (6) is $p(\lambda) = m\lambda^2 + b\lambda + k$. Homogenous solutions depend on the roots of the characteristic polynomial. If λ_1, λ_2 are the roots of p:

Roots	Homogenous	Solutions	Discriminant	Spring Case
$\lambda_1 \neq \lambda_2 \in \mathbb{R}$	$e^{\lambda_1 t}$	$e^{\lambda_2 t}$	$b^2 < 4mk$	Overdamped
$\lambda_1 = \lambda_2 \in \mathbb{R}$	$e^{\lambda_1 t}$	$te^{\lambda_2 t}$	$b^2 = 4mk$	Critically damped
$\lambda_\pm = \alpha \pm i\omega$	$e^{\alpha t}\sin\omega t$	$e^{\alpha t}\cos\omega t$	$b^2 > 4mk$	Underdamped
$\lambda_\pm = \pm i\omega$	$\sin\omega t$	$\cos\omega t$	$b = 0$	Undamped

3. Method of Undetermined Coefficients: Determine a particular homogenous solution by plugging the anzatz into the differential equation and attempting to fix the unknown constants. In general, if the inhomogeneity is of the form

$$p(t)e^{\lambda t}$$

for a polynomial p, then you should guess

$$t^m q(t) e^{\lambda t}$$

where m is the algebraic multiplicity of λ as an eigenvalue and q is a polynomial with unknown coefficients and $\deg q = \deg p$. Similarly

Inhomogeneity: $p(t)e^{\alpha t}\sin\omega t$

$p(t)e^{\alpha t}\cos\omega t$ Ansatz: $t^m e^{\alpha t} q(t)\sin\omega t + t^m e^{\alpha t}\tilde{q}(t)\cos\omega t$

where m is the algebraic multiplicity of $\alpha \pm \omega$ and $q, \tilde{q}$ are degree $\deg p$ polynomials of with unknown coefficients.

IX **Laplace Transform:** The Laplace Transform is a linear operator which acts on a function by f

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,dt$$

1. Linear: $\mathcal{L}\left[af(t) + bg(t)\right](s) = a\mathcal{L}[f(t)](s) + b\mathcal{L}[g(t)](s)$
2. Invertible: there is an inverse linear transform $\mathcal{L}^{-1}$ such that

$$\mathcal{L}^{-1}[\mathcal{L}\left[f(t)\right]] = f(t)$$

for any piecewise continuous, exponentially dominated function $f : [0, \infty) \to \mathbb{R}$.

3. Derivatives transform to multiplication by the frequency:

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right](s) = s\mathcal{L}\left[f(t)\right](s) - f(0)$$

4. Exponentials in time transform to shifts in frequency:

$$\mathcal{L}\left[e^{at}f(t)\right](s) = \mathcal{L}\left[f(t)\right](s-a)$$

5. Multiplication by time transforms to derivatives:

$$\mathcal{L}\left[tf(t)\right](s) = -\frac{d}{ds}\mathcal{L}\left[f(t)\right](s)$$

6. A piecewise continuous, exponentially dominated function satisfies:

$$\lim_{s\to\infty}\mathcal{L}\left[f(t)\right](s) = 0$$

7. Time dilation gives inverse frequency dilation

$$\mathcal{L}\left[f(at)\right](s) = \frac{1}{a}\mathcal{L}\left[f(t)\right]\left(\frac{s}{a}\right)$$

8. To solve a differential equation in independent variable y: Transform an differential equation in time t to an algebraic equation in terms of the Laplace variable s, then solve for $\mathcal{L}\left[y\right]$ in terms of s and invert the transform. Invert by means of a Laplace transform table (learn to use the table on page 328) and the method of partial fraction decompositions.
9. Write piecewise continuous functions using the *unit step* or *Heaviside function*

$$u_c(t) = u(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{if } t < c \end{cases}$$

which satisfies

$$\mathcal{L}\left[u(t-c)f(t-c)\right](s) = e^{-cs}\mathcal{L}\left[f(t)\right](s)$$

10. The impulse, point mass, or δ-Dirac function $\delta(t)$ may be thought of as the (distributional) derivative of the Heaviside function. It has the property:

$$\int_a^b \delta(t-t_0)f(t)dt = f(t_0)$$

if $t_0 \in (a,b)$ and 0 if $t_0 \notin [a,b]$. $\mathcal{L}\left[\delta(t)\right] = 1$.
11. A periodic function f with period T satisfies $f(t+T) = f(t)$ for any argument value t. If f is periodic then

$$\mathcal{L}\left[f\right](s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}} = \frac{1}{1-e^{-sT}}\mathcal{L}\left[f(t)\left(1-u(t-T)\right)\right](s)$$

12. The convolution of a functions f and g is written $f*g$ and defined by

$$f*g(t) = \int_0^t f(t-u)g(u)\,du$$

The convolution is commutative $f*g = g*f$ and linear in each argument.
13. Convolutions transform to multiplications $\mathcal{L}\left[f*g\right](s) = \mathcal{L}\left[f\right](s)\mathcal{L}\left[g\right](s)$

Laplace Transform $\mathcal{L}$

Time	**Frequency**	**Time**	**Frequency**
$f(t) = \mathcal{L}^{-1}[F](t)$	$\mathcal{L}[f](s) = F(s)$	$\mathcal{L}^{-1}[F](t) = f(t)$	$F(s) = \mathcal{L}[f](s)$
$f+g$	$\mathcal{L}[f]+\mathcal{L}[g]$	cf	$c\mathcal{L}[f]$
f'	$s\mathcal{L}[f]-f(0)$	$f^{(n)}$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$
$tf(t)$	$-\frac{d}{ds}F(s)$	$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$f(t) = f(t+T)$	$\frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$	$f*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$	$\mathcal{L}\left[f\right]\mathcal{L}\left[g\right]$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$\frac{1}{a}f\left(\frac{t}{a}\right)$	$F(as)$
1	$\frac{1}{s}$	$\delta(t-c)$	e^{-cs}
$e^{\lambda t}$	$\frac{1}{s-\lambda}$	$e^{\lambda t}f(t)$	$F(s-\lambda)$
t^n	$\frac{n!}{s^{n+1}}$	t^p for $p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh at$	$\frac{1}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$u(t-c)$	$\frac{e^{-cs}}{s}$	$u(t-c)f(t-c)$	$e^{-cs}\mathcal{L}[f(t)](s)$

Heaviside unit step function u, Dirac delta δ, gamma function Γ