Math 2552D
Spring 2017
Final Exam
2 May 2017
Time Limit: 2:50-5:40

Name: ________________________

This exam contains 13 pages (including this cover page) and 9 questions. There are 58 points in total. Write explanations clearly and in complete thoughts. No calculators may be used. Put your name on every page. You must **include units** on quantities that carry units. There is no need to simplify arithmetic expressions.

Grade Table

Question	Points	Score
1	10	
2	6	
3	9	
4	6	
5	6	
6	6	
7	6	
8	9	
9	0	
Total:	58	

Formal Symbols Crib Sheet

$f : A \to B$	function domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\mathbb{R}^n$	size n real vectors	$\mathbb{R}^{n\times m}$	size $n \times m$ real matrices
$\lim_{x\to a^-}$	limit from below	(a,b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a,b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x	$\mathcal{L}[f]$	Laplace transform of f

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f \circ g$	$(f' \circ g)g'$
f^{-1}	$\frac{1}{f' \circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x \rvert$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\text{arccsc} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\text{arcsec} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\text{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

Laplace Transform $\mathcal{L}$

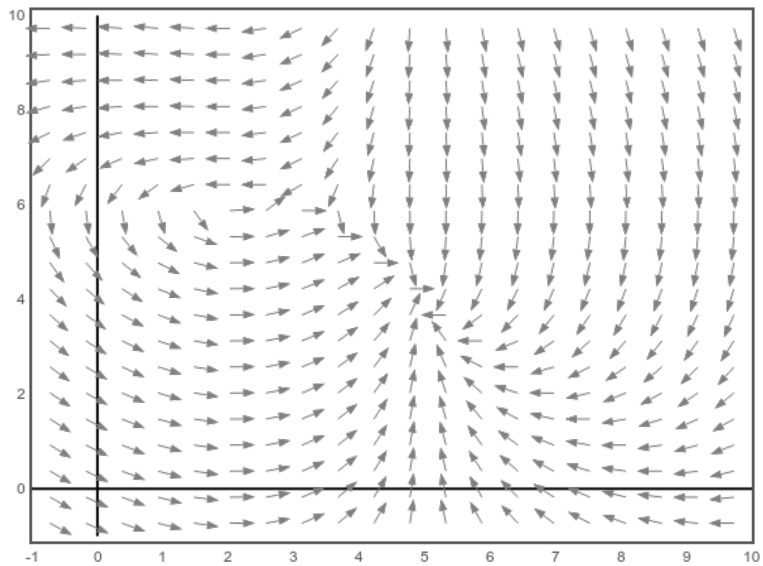
Time	**Frequency**	**Time**	**Frequency**
$f(t)$	$\mathcal{L}[f](s)$	$\mathcal{L}^{-1}[F](t)$	$F(s)$
$f+g$	$\mathcal{L}[f]+\mathcal{L}[g]$	cf	$c\mathcal{L}[f]$
f'	$s\mathcal{L}[f]-f(0)$	$f^{(n)}$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$
$tf(t)$	$-\frac{d}{ds}F(s)$	$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$f(t)=f(t+T)$	$\frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$	$f*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$	$\mathcal{L}[f]\,\mathcal{L}[g]$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$\frac{1}{a}f\left(\frac{t}{a}\right)$	$F(as)$
1	$\frac{1}{s}$	$\delta(t-c)$	e^{-cs}
$e^{\lambda t}$	$\frac{1}{s-\lambda}$	$e^{\lambda t}f(t)$	$F(s-\lambda)$
t^n	$\frac{n!}{s^{n+1}}$	t^p for $p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh at$	$\frac{1}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$u(t-c)$	$\frac{e^{-cs}}{s}$	$u(t-c)f(t-c)$	$e^{-cs}\mathcal{L}[f(t)](s)$

Heaviside unit step function u, Dirac delta δ, gamma function Γ

Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$

Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$

1. Consider the direction field for a two dimensional system of first order differential equations shown below.



(a) (3 points) Is the differential equation linear or nonlinear? How do you know?

Solution: The equation is nonlinear. It cannot be linear since it appears to have fixed points at $(5, 4)$ and $(3, 6)$ and the direction field is not parallel along subspaces.

(b) (3 points) Suppose that the differential equation above is given initial condition $\begin{pmatrix} 9 \\ 8 \end{pmatrix}$. Estimate $\lim_{t\to\infty} \mathbf{x}(t)$.

Solution: We can estimate

$$\lim_{t\to\infty} \mathbf{x}(t) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

by tracing curves tangent to the direction field.

(c) (4 points) Which of the following are possibly problems with the numerical solution $x(t) = \phi(t)$ to a differential equation $\dot{x} = f(x, t)$ with initial condition $x(t_0) = x_0$ estimated via Euler's method? Circle ALL the true statements.

A. If the equation is chaotic, then small changes in small round off errors might result in radically different results.

B. If the second derivative ϕ'' is bounded, the local truncation error may grow exponentially in the step size.

C. If the step size is too large, it is possible that the numerical solution could erroneously cross a separatrix in phase space.

D. If f or $\frac{\partial f}{\partial x}$ have a discontinuity at $t_1 > t_0$, there may not be a unique solution for $t \geq t_1$.

Solution: Statements A, C, and D are true. Statement B is false since for step size h the local truncation error is bounded by $\frac{1}{2}|\phi''|h^2$. So the local truncation error grows at most quadratically in the step size.

2. (6 points) Find the general solution to the differential equation by any method.

$$y'' - 2y' + y = 6t$$

Solution: The characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. So we have the homogeneous part of the solution set is spanned by e^t and te^t. To find an inhomogenous solution we use the method of undetermined coefficients. Since the inhomogeneity is a first degree polynomial, we guess a solution of the form $y_p = At + B$. Then

$$y_p'' - 2y_p' + y_p = At + B - 2A = 6t$$

forces $A = 6$ and $B = 12$. Then general solution is thus

$$y = c_1 e^t + c_2 te^t + 6t + 12$$

for any $c_1, c_2 \in \mathbb{R}$.

3. (9 points)

$$A = \begin{pmatrix} -13 & 25 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

(a) Compute the matrix exponential e^{At}.

Solution: From the Jordan Form of the matrix above we see that there three linearly independent solutions

$$x_1(t) = e^{-3t} \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$$

$$x_2(t) = e^{-3t} \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \right)$$

$$x_3(t) = e^{4t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so that we can form fundamental matrix

$$X(t) = \begin{pmatrix} 5e^{-3t} & e^{-3t}(2+5t) & 0 \\ 2e^{-3t} & e^{-3t}(1+2t) & 0 \\ 0 & 0 & e^{4t} \end{pmatrix}$$

whose columns span the solution space. Then the matrix exponential is

$$\begin{aligned} e^{At} = X(t)X^{-1}(0) &= \begin{pmatrix} 5e^{-3t} & e^{-3t}(2+5t) & 0 \\ 2e^{-3t} & e^{-3t}(1+2t) & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5e^{-3t} & e^{-3t}(2+5t) & 0 \\ 2e^{-3t} & e^{-3t}(1+2t) & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \frac{1}{1} \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (1-10t)e^{-3t} & 25te^{-3t} & 0 \\ -4te^{-3t} & (1+10t)e^{-3t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \end{aligned}$$

(b) Compute the solution to the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ if $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$.

Solution:

$$\mathbf{x}(t) = e^{At}\begin{pmatrix}5\\2\\2\end{pmatrix} = \begin{pmatrix}5e^{-3t}\\2e^{-3t}\\2e^{4t}\end{pmatrix}$$

4. (6 points) Choose one of the following 3 differential equations below and give the general solution. Circle the equation you are solving.

   1. $\frac{dz}{dt} + \frac{z}{t} = \ln|t|$
   2. $\frac{dw}{dt} = w(1-3w)$
   3. $(\ln|tv|+1)\frac{dv}{dt} = 1 - \frac{v}{t}$

Solution: 1.) The equation is linear so we multiply through by integrating factor $e^{\int \frac{dt}{t}} = t$ to obtain $t\ln|t| = t\frac{dz}{dt} + z = \frac{d}{dt}(tz)$ so that $z = \frac{1}{t}\int t\ln|t|dt$ integrate by parts to obtain

$$z = \frac{t}{2}\ln|t| - \frac{t}{4} + \frac{c}{t}$$

for any constant $c \in \mathbb{R}$.

2.) The equation is separable so we have either $w = 0$ or $1/3$ or $\int \frac{dw}{w(1-3w)} = \int dt = t + c$. We can integrate by using partial fractions

$$\int \frac{dw}{w(1-3w)} = \int \frac{1}{w} + \frac{3}{1-3w}\, dw = \ln|w| - \ln|1-3w| + c_0 = \ln\left|\frac{w}{1-3w}\right| + c_0$$

Then setting $A = \pm e^{c-c_0}$ we have $\frac{w}{1-3w} = Ae^t$ and we solve for w to find the general solution

$$w(t) = \frac{Ae^t}{1+3Ae^t} \text{ or } \frac{1}{3}$$

for any $A \in \mathbb{R}$.

3.) The equation is exact. Suppose that the solutions are the level sets of a potential function ψ. We have that $\frac{\partial}{\partial t}\psi = \frac{v}{t} - 1$. Then $\psi(v,t) = v\ln t - t + g(v)$ for some function g of v only. Comparing two expressions for $\frac{\partial}{\partial v}\psi$ we can solve for g.

$$g'(v) = \frac{\partial\psi}{\partial v} - \ln|t| = \ln|vt| + 1 - \ln|t| = \ln|v|$$

so that $g(v) = v\ln|v| - v$ and the general solution is

$$c = \psi(v,t) = v\ln|vt| - t - v$$

for any constant c.

5. (6 points) Consider the periodic function $g : [0, \infty) \to \mathbb{R}$ with period 4 given by

$$g(t) = \begin{cases} e^{-5t} & \text{if } t \in [0, 4) \\ g(t-4n) & \text{if } t \in [4n, 4n+4) \text{ for any positive integer n} \end{cases}$$

Compute the Laplace transform $\mathcal{L}[g]$.

Solution: Use the Laplace periodicity rule.

$$\begin{aligned} \mathcal{L}[g](s) &= \frac{\int_0^4 e^{-5t} e^{-st} dt}{1 - e^{-4s}} \\ &= \frac{\int_0^4 e^{-(s+5)t} dt}{1 - e^{-4s}} \\ &= \frac{-e^{-(s+5)t}|_{t=0}^4}{(s+5)(1 - e^{-4s})} \\ &= \frac{1 - e^{-4(s+5)}}{(s+5)(1 - e^{-4s})} \end{aligned}$$

6. (6 points) Compute the inverse Laplace transform $\mathcal{L}^{-1}[H]$ of the frequency function

$$H(s) = 4 + \frac{2e^{-s}}{(s-3)^3}$$

Solution: Use the Laplace rule for the Dirac delta function, the unit step function, and multiplication by exponentials.

$$\begin{aligned}
\mathcal{L}^{-1}\left[H\right](t) &= 4\delta(t) + \mathcal{L}^{-1}\left[\frac{2e^{-s}}{(s-3)^3}\right](t) \\
&= 4\delta(t) + u(t-1)\mathcal{L}^{-1}\left[\frac{2}{(s-3)^3}\right](t-1) \\
&= 4\delta(t) + u(t-1)e^{-3(t-1)}\mathcal{L}^{-1}\left[\frac{2}{s^3}\right](t-1) \\
&= 4\delta(t) + u(t-1)e^{-3(t-1)}(t-1)^2
\end{aligned}$$

7. (6 points) An investment account is opened with \$1000 on Jan 1, 2017. The account pays an annual 5% continuously compounded interest rate. Continuous withdrawals are made at a rate of \$100 per year. A large deposit of \$3,000 is made on Jan 1, 2020. Write an initial value problem to model the value of the account at all time after it is opened. What are the quantities described by all of your variables and what units do they carry? You do not need to solve your equation, only explain it.

Solution: Let $S(t)$ be the value of the account at time t measured in years from Jan 1, 2017. We have that $S(0) = 1000$ and the derivative is given by

$$\frac{dS}{dt} = .05S - 100 + 3000\delta(t-3)$$

8. (9 points) An undamped spring with a 2kg mass and a 8N/m Hooke coefficient is initially stretched 1 meter up from equilibrium then released with zero initial velocity. After 30 seconds, an exponentially decaying force of $16e^{-2(t-30)}$ N is applied to the mass at time $t \geq 30$. Compute the position of the spring (as measured from the equilibrium) for all positive time. (Hint: You may find the partial fraction decomposition below helpful.)

$$\frac{1}{(s+a)(s^2+b^2)} = \frac{1}{a^2+b^2}\frac{1}{s+a} + \frac{1}{a^2+b^2}\frac{-s+a}{s^2+b^2}$$

Solution: Let $y(t)$ be the displacement of the mass from equilibrium at time t. The equation of motion is

$$2y'' + 8y = 16e^{-2(t-30)}u(t-30)$$

with $y(0) = 1$ and $y'(0) = 0$. Divide out the common 2 and take the Laplace transformation of both sides.

$$\begin{aligned}
\mathcal{L}\left[y'' + 4y\right] &= \mathcal{L}\left[8e^{-2(t-30)}u(t-30)\right] \\
s^2\mathcal{L}\left[y\right](s) - s + 4\mathcal{L}\left[y\right](s) &= 8e^{-30s}\mathcal{L}\left[e^{-2t}\right](s) \\
(s^2+4)\mathcal{L}\left[y\right](s) - s &= 8e^{-30s}\frac{1}{s+2} \\
\mathcal{L}\left[y\right](s) &= \frac{s}{s^2+4} + e^{-30s}\frac{8}{(s+2)(s^2+4)} \\
&= \frac{s}{s^2+4} + e^{-30s}\left(\frac{1}{s+2} + \frac{-s+2}{s^2+4}\right) \\
&= \mathcal{L}[\cos 2t](s) + e^{-30s}\mathcal{L}\left[e^{-2t} - \cos 2t + \sin 2t\right](s) \\
&= \mathcal{L}\left[\cos 2t + u(t-30)(e^{-2(t-30)} - \cos 2(t-30) + \sin 2(t-30))\right](s)
\end{aligned}$$

so

$$y(t) = \cos 2t + u(t-30)\left(e^{-2(t-30)} - \cos 2(t-30) + \sin 2(t-30)\right)$$

9. BONUS: Name one of the mathematicians whose ideas we discussed this semester. What was their idea?