Math 2552D
Spring 2017
Exam 1
9 Feb 2017
Time Limit: 80 Minutes

Name: Key

This exam contains 8 pages (including this cover page) and 6 questions. There are 55 points in total. Write explanations clearly and in complete thoughts. No calculators may be used. Put your name on every page. You must **include units** on quantities that carry units. There is no need to simplify arithmetic expressions.

Grade Table

Question	Points	Score
1	8	
2	12	
3	12	
4	8	
5	9	
6	6	
Total:	55	

Formal Symbols Crib Sheet

$f : A \to B$	function with domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\lim_{x\to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x		

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f \circ g$	$(f' \circ g)g'$
f^{-1}	$\frac{1}{f' \circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x\rvert$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arccsc} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\operatorname{arcsec} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

Geometry Crib Sheet

Pythagorean Identity $a^2+b^2=c^2$

Circle: radius r	$A=\pi r^2$	$c=2\pi r$
Box: dimensions x, y, z	$V=xyz$	$A=2(yz+xz+xy)$
Sphere: radius r	$V=\frac{4}{3}\pi r^3$	$A=4\pi r^2$
Right pyramid: height h dim x,y	$V=\frac{1}{3}hxy$	$A=xy+x\sqrt{(y/2)^2+h^2}+y\sqrt{(x/2)^2+h^2}$
Cylinder: height h radius r	$V=\pi hr^2$	$A=2\pi r(h+r)$
Right Cone: height h radius r	$V=\frac{\pi}{3}hr^2$	$A=\pi r\left(r+\sqrt{r^2+h^2}\right)$
Torus: radii $R>r$	$V=2\pi^2r^2R$	$A=4\pi^2rR$
Tetrahedron: edge x	$V=\frac{1}{6\sqrt{2}}x^3$	$A=\sqrt{3}x^2$
Octahedron: edge x	$V=\frac{\sqrt{2}}{3}x^3$	$A=2\sqrt{3}x^2$
Dodecahedron: edge x	$V=\frac{15+7\sqrt{5}}{4}x^3$	$A=3\sqrt{20+10\sqrt{5}}x^2$
Icosahedron: edge x	$V=\frac{5\left(3+\sqrt{5}\right)}{12}x^3$	$A=5\sqrt{3}x^2$

1. (a) (2 points) How many solutions can a first order differential equation have if an initial condition is specified?

   A. infinity many
   B. finitely many
   C. exactly one
   D. none
   E. any of the above

   (b) (3 points) For what points (t_0, y_0) is the initial value problem $y(t_0) = y_0$ and

   $$y' = (1 - t^2 - y^2)^{\frac{1}{2}}$$

   guaranteed to have a unique solution?

   $\frac{\partial f}{\partial y} = y\,(1-t^2-y^2)^{-1/2}$

   $\Rightarrow$ f and $\frac{\partial f}{\partial y}$ are continuous if

   $$\boxed{t_0^2 + y_0^2 < 1}$$

   (c) (3 points) What type of equation is eqution 1? Circle the correct descriptors below.

   $$y'' + t^3 y' = y \sin t + 10 \tag{1}$$

   (a) Equation 1 has order 2
   (b) LINEAR NONLINEAR
   (c) AUTONOMOUS NONAUTONOMOUS
   (d) HOMOGENOUS INHOMOGENOUS with inhomogeneity 10

2. (a) (3 points) Is equation 2 exact? Why or why not?

$$t\sin\left(y^2+t^2\right)+\left(t+y\sin\left(y^2+t^2\right)\right)y'=-y \tag{2}$$

$\frac{\partial}{\partial y}\left(t\sin(y^2+t^2)+y\right) = 1+2yt\cos(t^2+y^2)$

$\frac{\partial}{\partial t}\left(t+y\sin(y^2+t^2)\right) = 1+2yt\cos(t^2+y^2)$

Yes! It is exact since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$

(b) (3 points) Equation 3 is exact. Find a quantity conserved by solutions to equation 3.

$$2yx+\frac{y}{x^2}+\left(x^2-\frac{1}{x}\right)\frac{dy}{dx}=0 \tag{3}$$

$\psi(x,y) = yx^2 - \frac{y}{x}$

(c) (3 points) Give the general solution to equation 3.

$y = \frac{C}{x^2 - 1/x}$ for some $C \in \mathbb{R}$

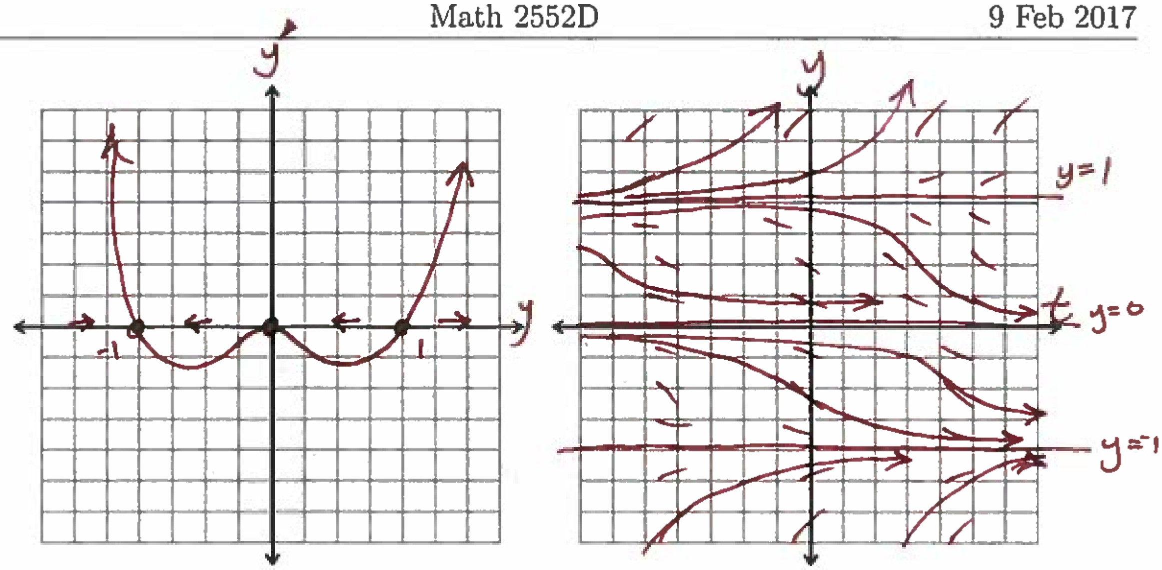
(d) (3 points) Equation 4 can be made exact after multiplying by an integrating factor. What integrating factor should be used?

$$t^2e^y+\left(t^3e^y+t^2\right)y'=0 \tag{4}$$

$\dot{\mu} = \frac{t^2e^y-(3t^2e^y+2t)}{t^3e^y+t^2}\mu = \frac{-2}{t}\mu$

$\Rightarrow \int\frac{d\mu}{\mu} = -2\int\frac{dt}{t} \Rightarrow \ln|\mu| = \ln|t^{-2}|+c$

Use integrating factor $\mu = t^{-2}$



3.

(a) (3 points) Sketch the phase portrait in the y-y' plane for the equation $y' = y^4 - y^2$.

(b) (3 points) Sketch a slope field and some solutions to the equation $y' = y^4 - y^2$.

(c) (3 points) What are the equilibrium points for y? What is the stability of each point?

Equilibrium:

$y = 1$ unstable

$y = 0$ semistable

$y = -1$ stable

(d) (3 points) Describe the long term behavior of solutions in terms of the initial condition $y(0) = y_0$.

$$\lim_{t \to \infty} y(t) = \begin{cases} \infty & \text{if } y_0 > 1 \\ 1 & \text{if } y_0 = 1 \\ 0 & \text{if } 1 > y_0 \geq 0 \\ -1 & \text{if } y_0 < 0 \end{cases}$$

4. A 100 gallon tank is cylindrically shaped with . The tank starts with 50 gallons of water and 3 lbs of salt. 2 gallons per minute of saltwater at a concentration of 0.1 lb/gal. When the tank is full, a pressure valve allows 2 gallons per minute of mixed fluid out.

   (a) (3 points) Set up a differential equation governing the concentration of salt in the tank from time 0 until 25 minutes.

$V(t) = 50 + 2t$

$s = Q \cdot V$

$\frac{ds}{dt} = 0.2 \text{ lb/min}$

$\Rightarrow s' = Q'V + V'Q$

$\Rightarrow 0.2 \text{ lb/min} = Q'(50+2t)\text{gal} + 2\frac{\text{gal}}{\text{min}} Q$

$$\Rightarrow \boxed{\frac{dQ}{dt} = \frac{-2}{50+2t} Q + \frac{0.2}{50+2t} \quad \text{lb/gal min}}$$

   (b) (2 points) Give an initial condition at time 0 for the differential equation.

$$\boxed{Q(0) = \frac{3}{50} \frac{\text{lb}}{\text{gal}}}$$

   (c) (3 points) Set up a differential equation governing the concentration of salt in the tank after 25 minutes.

$\frac{ds}{dt} = 0.2 \text{ lb/min} - 2Q \frac{\text{gal}}{\text{min}}$

$s = 100Q$

$\Rightarrow 100\text{gal} \frac{dQ}{dt} = 0.2 \text{ lb/min} - 2\frac{\text{gal}}{\text{min}} \cdot Q$

$$\boxed{\frac{dQ}{dt} = \frac{0.2}{100} - \frac{2}{100} Q \quad \text{lb/gal min}}$$

5. (a) (3 points) Use Euler's method with step size $h = 1$ to estimate $y(2)$ if $y(1) = 3$ and $y' = t^2 + y$. Be sure to make each piece of your calculation clear!

t	y	y'
1	3	1^2+3 4
2	$3+1\cdot 4$ 7	2^2+7 11

$$\boxed{y_{\text{Euler}}(2) = 7}$$

(b) (3 points) Use Euler's improved method to obtain a better estimation of $y(2)$

Improved slope $\frac{4+11}{2} = \frac{15}{2}$

$$y_{\text{Improved}}(2) = 3 + 1\cdot\frac{15}{2} = \boxed{10.5}$$

(c) (3 points) Suppose that a differential equation $y' = f(y, t)$ is difficult to solve, but we are sure that $|y| < 10$ and $|y'| < 20$ and $|y''| < 30$. What step size h should be used to make sure the local trucation error is less than $\frac{1}{100}$?

$$\frac{h^2}{2}\cdot 30 \leqslant \frac{1}{100}$$

$$\Rightarrow h \leqslant \sqrt{\frac{2}{3000}}$$

6. (a) (3 points) Find the implicit general solution to

$$\frac{dy}{dt} = \frac{t^2}{1-y^2}$$

Separable $\int (1-y^2)\,dy = \int t^2\,dt$

$$\boxed{y - \frac{1}{3}y^3 = \frac{1}{3}t^3 + C}$$

(b) (3 points) Solve the initial value problem with $y(0) = 1$.

$$1 - \frac{1}{3}\cdot 1^3 = \frac{1}{3}\cdot 0^3 + C$$

$$\Rightarrow C = \frac{2}{3}$$

$$\boxed{y - \frac{1}{3}y^3 = \frac{1}{3}t^3 + \frac{2}{3}}$$