Math 2552D
Spring 2017
Exam 2
30 March 2017
Time Limit: 80 Minutes

KEY

Name: SOLUTIONS

This exam contains 7 pages (including this cover page) and 6 questions. There are 52 points in total. Write explanations clearly and in complete thoughts. No calculators may be used. Put your name on every page. You must **include units** on quantities that carry units. There is no need to simplify arithmetic expressions.

Grade Table

Question	Points	Score
1	10	
2	7	
3	8	
4	8	
5	8	
6	11	
Total:	52	

Formal Symbols Crib Sheet

$f : A \to B$	function with domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x \to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\mathbb{R}^n$	size n real vectors	$\mathbb{R}^{n \times m}$	size $n \times m$ real matrices
$\lim_{x \to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x \to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x		

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f\circ g$	$(f'\circ g)g'$
f^{-1}	$\frac{1}{f'\circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x\rvert$	$\frac{1}{x\ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\mathrm{arccsc} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\mathrm{arcsec} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\mathrm{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

Geometry Crib Sheet

Pythagorean Identity $\sin^2(t)+\cos^2(t)=1$

Circle: radius r	$A=\pi r^2$	$c=2\pi r$
Box: dimensions x, y, z	$V=xyz$	$A=2(yz+xz+xy)$
Sphere: radius r	$V=\frac{4}{3}\pi r^3$	$A=4\pi r^2$
Right pyramid: height h dim x,y	$V=\frac{1}{3}hxy$	$A=xy+x\sqrt{(y/2)^2+h^2}+y\sqrt{(x/2)^2+h^2}$
Cylinder: height h radius r	$V=\pi hr^2$	$A=2\pi r(h+r)$
Right Cone: height h radius r	$V=\frac{\pi}{3}hr^2$	$A=\pi r\left(r+\sqrt{r^2+h^2}\right)$
Torus: radii $R>r$	$V=2\pi^2r^2R$	$A=4\pi^2rR$
Tetrahedron: edge x	$V=\frac{1}{6\sqrt{2}}x^3$	$A=\sqrt{3}x^2$
Octahedron: edge x	$V=\frac{\sqrt{2}}{3}x^3$	$A=2\sqrt{3}x^2$
Dodecahedron: edge x	$V=\frac{15+7\sqrt{5}}{4}x^3$	$A=3\sqrt{20+10\sqrt{5}}x^2$
Icosahedron: edge x	$V=\frac{5\left(3+\sqrt{5}\right)}{12}x^3$	$A=5\sqrt{3}x^2$

1. (10 points) Are the following statements about Euler's method true of a linear differential equation? Are they true for a chaotic differential equation like the Lorenz equation? Circle if the statement is true or false for a linear equation and for a chaotic equation.

   (a) Euler's method could be used to estimate the location of an attracting fixed point or a strange attractor.

   True/False for a linear equation. True/False for a chaotic equation.

   (b) Euler's method could be used to estimate the location of separatrices dividing the basins of multiple attractors.

   True/False for a linear equation. True/False for a chaotic equation.

   a linear equation cannot have multiple basins of attraction

   (c) Euler's method will become more accurate for more steps/if the step size is reduced.

   True/False for a linear equation. True/False for a chaotic equation.

   (d) Euler's method will yield different, random results each time it is computed with the same input.

   True/False for a linear equation. True/False for a chaotic equation.

   The method is deterministic.

   (e) Two different computers that round to a different number of digits during Euler's method might produce radically different results with the same initial condition and step size.

   True/False for a linear equation. True/False for a chaotic equation.

2. (7 points) Suppose that $\frac{d}{dt}\mathbf{x} = F(\mathbf{x})$ is a 2 dimensional nonlinear differential equation with a fixed point $\mathbf{a}$. Suppose further that $F'(\mathbf{a})$ has a negative determinant. What are the possibilities for the stability of the fixed point $\mathbf{a}$? Circle all that apply.

   A. Attracting node
   B. Semistable saddle point
   C. Repelling node
   D. Attracting spiral point
   E. Center spiral point
   F. Repelling spiral point
   G. Degenerate sheer node

Since $\det F'(a) = \lambda_1 \cdot \lambda_2 < 0$ it must be that $\lambda_2 > 0 > \lambda_1$ are distinct, real, and have different signs

3. (8 points) Solve the initial value problem for $y(t)$ if

$$y'' - 4y' + 20y = 0$$

and $y(0) = 5$ and $y'(0) = 2$.

The characteristic polynomial is $\lambda^2 - 4\lambda + 20 = (\lambda-2)^2 + 16 \Rightarrow \lambda = 2 \pm 4i$

So the general solution is $y(t) = C_+ e^{(2+4i)t} + C_- e^{(2-4i)t}$ for $C_+, C_- \in \mathbb{C}$

or $= e^{2t}(C_1 \cos 4t + C_2 \sin 4t)$ for $C_1, C_2 \in \mathbb{R}$

Then $y(0) = 5 = C_1$ and $y' = 2e^{2t}(C_1 \cos 4t + C_2 \sin 4t) + e^{2t}(-4C_1 \sin 4t + 4C_2 \cos 4t)$

$= e^{2t}\left((2C_1 + 4C_2)\cos 4t + (4C_2 - 2C_1)\sin 4t\right)$

$\Rightarrow y'(0) = 2 = 2C_1 + 4C_2 \Longrightarrow C_2 = \frac{2-10}{4} = -2$

$y(t) = e^{2t}(5\cos 4t - 2\sin 4t)$

Alternatively $\begin{pmatrix} 1 & 1 \\ 2+4i & 2-4i \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 5 \\ 2+4i & 2-4i & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 5 \\ 0 & -8i & -8-20i \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -i+5/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & +5/2+i \\ 0 & 1 & 5/2-i \end{pmatrix}$

$y(t) = \left(\frac{5}{2} + i\right)e^{(2+4i)t} + \left(\frac{5}{2} - i\right)e^{(2-4i)t}$

4. (8 points) Find four linearly independent solutions to the diffeq $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ if A is the matrix below.

$$\begin{pmatrix} -7 & 0 & 0 & 6 \\ -1 & -1 & -1 & 10 \\ 3 & -18 & -4 & -27 \\ 1 & -6 & 1 & -16 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 1 & 0 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 & 6 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1}$$

From the Jordan form:

$$x_1(t) = e^{-7t}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$x_2(t) = e^{-7t}\begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_3(t) = e^{-7t}\left(\begin{pmatrix} 0 \\ -1 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$x_4(t) = e^{-7t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 3 \\ 1 \end{pmatrix} + \frac{t}{2}\begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

Many other bases for the solution space are valid.

5. (8 points) Compute the matrix exponential e^{Bt} if

$$B = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$$

the characteristic polynomial is $(1-\lambda)(-1-\lambda)-8 = \lambda^2-9 \Rightarrow \lambda = \pm 3$

for $\lambda = 3$ $\quad$ null$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}$ is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

for $\lambda = -3$ $\quad$ null$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}$ is spanned by $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

so

$$e^{Bt} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2e^{3t} & e^{-3t} \\ e^{3t} & -e^{-3t} \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2e^{3t}+e^{-3t} & 2e^{3t}-2e^{-3t} \\ e^{3t}-e^{-3t} & e^{3t}+2e^{-3t} \end{pmatrix}$$

6. Consider the following system of nonlinear differential equations.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}x(y-1) \\ 2\,(x-3)y \end{pmatrix}$$

(a) (2 points) Find the critical points of the system.

If $x=0$ then $y=0$. If $x \neq 0$ then $y=1$ then $x=3$.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(b) (3 points) At one of the critical points **a**, both x and y are nonzero. Write the linear differential equation which best approximates the system near the point **a**.

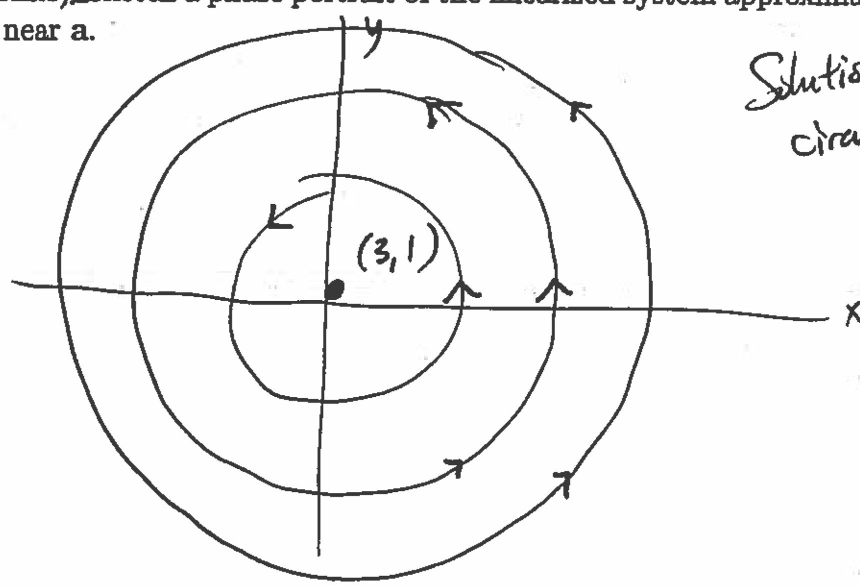
$$F'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2/3(y-1) & -2/3x \\ 2y & 2(x-3) \end{pmatrix} \quad \text{so} \quad F'\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

then near $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ we have $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x-3 \\ y-1 \end{pmatrix}$

(c) (3 points) Evaluate the stability of the critical point **a** [redacted]

The eigenvalues are purely imaginary. Solutions oscillate around $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. We cannot conclude if it is attracting or repelling. Appears to be a spiral center. (In fact, solutions are periodic)

(d) (3 points) Sketch a phase portrait of the linearized system approximating the equation near **a**.



Solutions have circular orbits