1. What behaviors can solutions to autonomous nonlinear differential equations have that do not exist in solutions to autonomous linear differential equations? In what ways are fixed points of the two systems different? In what ways are they the same?

2. Why might you not want to use Euler's method to numerically solve a chaotic nonlinear differential equation?

3. Suppose that $\frac{d}{dt}\mathbf{x} = F(\mathbf{x})$ is a 3 dimensional nonlinear differential equation with a fixed point $\mathbf{a}$. Suppose further that $F'(\mathbf{a})$ has a negative determinant. What are the possibilities for the stability of the fixed point $\mathbf{a}$?

4. Solve the initial value problem for $y(t)$ if
$$y'' - 5y' + 6y = 0$$
and $y(0) = 2$ and $y'(0) = -5$. Draw a phase portrait in y-y' space and sketch the solution in t-y space.

5. Verify that $X(t)$ is a fundamental matrix for the equation $\frac{d}{dt}\mathbf{x} = A(t)\mathbf{x}$ where
$$A(t) = \begin{pmatrix} \frac{2t}{t^2-1} & \frac{-2}{t^2-1} \\ 0 & \frac{1}{t} \end{pmatrix}$$
and
$$X(t) = \begin{pmatrix} t^2 & 1 \\ t & t \end{pmatrix}$$
What is the Wronskian for the set of solutions? Suppose we have initial condition $\mathbf{x}(2) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$. What is $\mathbf{x}(t)$? For what values of t is your solution valid?

6. The matrices below are not diagonalizable. Use the given factorization to find four linearly independent solutions to the diffeq $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ if A is one of the matrices below.

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -3 & 0 & 0 & 2 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -4 & 1 & 0 & 4 \\ 0 & -4 & 0 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -6 & 1 & 0 & 1 \\ -1 & -1 & -1 & 8 \\ 3 & -16 & -3 & -25 \\ 1 & -5 & 1 & -14 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -6 & 1 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -8 & 1 & 0 & 8 \\ -1 & -1 & -1 & 11 \\ 3 & -22 & -5 & -31 \\ 1 & -7 & 1 & -18 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -8 & 1 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & -8 & 1 \\ 0 & 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} 1 & 7 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1}$$

7. Compute the matrix exponential e^{At} if A is one of the following matrices:

$$\begin{pmatrix} -10 & 6 \\ -15 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ -5 & 6 \end{pmatrix}$$

8. Find the critical points for the following nonlinear systems. Evaluate the stability of the critical point **a** where both x and y are nonzero. Draw a phase portrait of the linearized system approximating the equation near **a**.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{5}(4\,x + 3\,y - 22)x \\ \frac{1}{30}(2\,x - 11\,y + 64)y \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -(5\,x - 4\,y + 11)x \\ -\frac{1}{4}(4\,x - 5\,y + 16)y \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{6}(x - 2\,y + 4)x \\ \frac{1}{9}(4\,x + y - 11)y \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{15}(7\,x + 10\,y - 45)x \\ \frac{1}{3}(4\,x - 11\,y - 9)y \end{pmatrix}$$

9. The following systems are written in polar coordinates. Find all the periodic solutions and classify the stability of their orbits and their basins of attraction. Draw a phase portrait of the solutions.

$$\frac{dr}{dt} = r(r-1)(r-4)^2$$
$$\frac{d\theta}{dt} = -2$$

$$\frac{dr}{dt} = r^4 - 6r^3 + 11r^2 - 6r$$
$$\frac{d\theta}{dt} = 4$$