Math 2552D
Spring 2017
Practice Final Exam
2 May 2017
Time Limit: 2:50-5:40

Name: ______________________

This practice exam contains 6 pages (including this cover page) and 13 questions. Write explanations clearly and in complete thoughts. No calculators may be used on the actual exam, but this practice exam contains more difficult calculations, and you will need a calculator. You may use a one page, two sided crib sheet. Put your name on every page. You must **include units** on quantities that carry units. There is no need to simplify arithmetic expressions.

Grade Table

Formal Symbols Crib Sheet

$f : A \to B$	function domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\mathbb{R}^n$	size n real vectors	$\mathbb{R}^{n\times m}$	size $n \times m$ real matrices
$\lim_{x\to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x	$\mathcal{L}[f]$	Laplace transform of f

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f \circ g$	$(f' \circ g)g'$
f^{-1}	$\frac{1}{f' \circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x \rvert$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\text{arccsc} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\text{arcsec} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\text{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

Laplace Transform $\mathcal{L}$

Time	**Frequency**	**Time**	**Frequency**
$f(t)$	$\mathcal{L}[f](s)$	$\mathcal{L}^{-1}[F](t)$	$F(s)$
$f+g$	$\mathcal{L}[f]+\mathcal{L}[g]$	cf	$c\mathcal{L}[f]$
f'	$s\mathcal{L}[f]-f(0)$	$f^{(n)}$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$
$tf(t)$	$-\frac{d}{ds}F(s)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f(t)=f(t+T)$	$\frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$	$f*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$	$\mathcal{L}[f]\,\mathcal{L}[g]$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$\frac{1}{a}f\left(\frac{t}{a}\right)$	$F(as)$
1	$\frac{1}{s}$	$\delta(t-c)$	e^{-cs}
$e^{\lambda t}$	$\frac{1}{s-\lambda}$	$e^{\lambda t}f(t)$	$F(s-\lambda)$
t^n	$\frac{n!}{s^{n+1}}$	t^p for $p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh at$	$\frac{1}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$u(t-c)$	$\frac{e^{-cs}}{s}$	$u(t-c)f(t-c)$	$e^{-cs}\mathcal{L}[f(t)](s)$

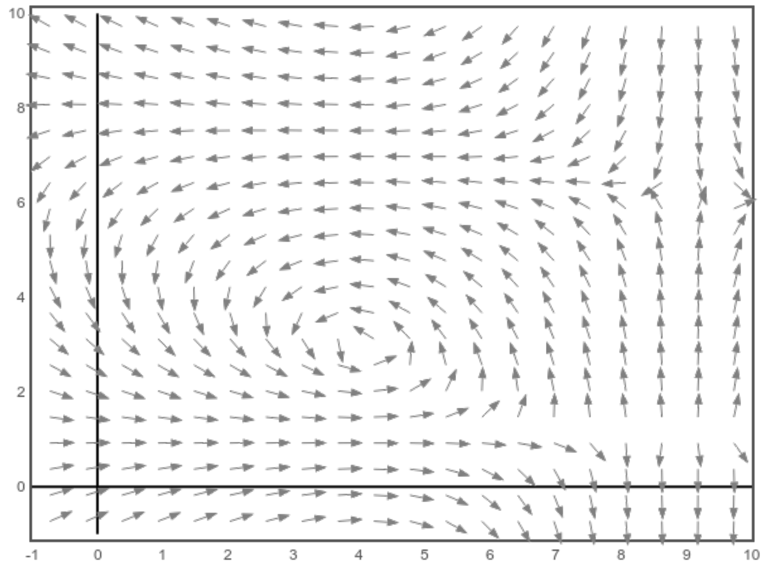
Heaviside unit step function u, Dirac delta δ, gamma function Γ

Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$

Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$

1. Consider the direction field for a two dimensional first order differential equation shown below.

$$\frac{d}{dt}\mathbf{x} = F(\mathbf{x})$$



What type of equation is the differential equation? Do you see evidence of chaos? What even is chaos again? Would a direction field like this show chaos? Do you see evidence of periodic orbits? Where are the fixed points? Classify their stabilities. Get out some colors. Sketch the basin of attraction for any stable fixed points. Draw the separatrices. Determine the limiting possible limiting values of $\mathbf{x}(t)$.

2. Vectorize the differential equation into a first order system, then use Euler method with step size 1/2 to estimate the value of $y(5)$ if $y(3) = -1$ and $y'(3) = 2$ and y satisfies the differential equation

$$y'' - \sqrt{t}y' + y = t$$

(Practice with a calculator or computer; the numbers will not be pretty.)

3. Use Euler's improved method with a step size of 1/4 to compute $y(3)$ if $y(2) = -1$ and y satisfies the differential equation

$$y' = y(y - t)$$

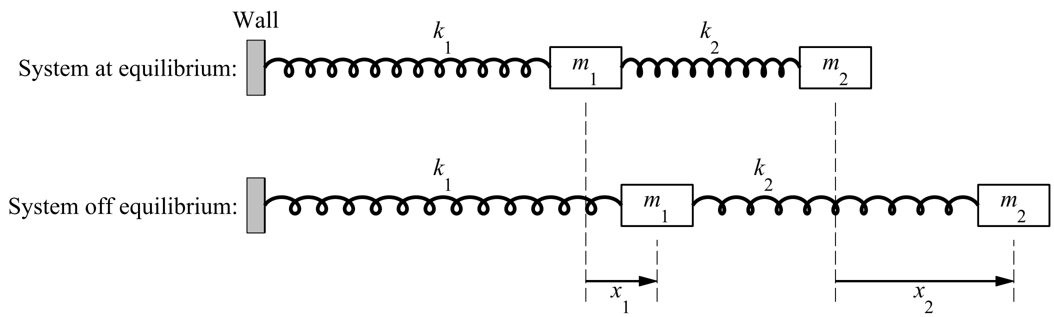
(Practice with a calculator or computer; the numbers will not be pretty.

4. Write the linear system that best approximates the differential equation

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a + 3a^2 - 12ab - b \\ a - 12ab^2 + 3b \end{pmatrix}$$

near its equilibrium point at the origin. Classify the stability. What are the approximate solutions near the origin? Draw a phase portrait for solutions near the origin.

5. Consider the two springs shown in the figure below. Spring 1 is attached to an immovable wall and a mass of $m_1 = 2$ kg and has a Hooke constant of $k_1 = 4$ N/m. Spring 2 is attached to mass 1 and a second mass of $m_2 = 1$ kg and has a Hooke constant of $k_2 = 4$ N/m. Both springs are undamped. Write a system of differential equations governing the motion of the springs. Vectorize your system to give a first order linear system. What are the resonant frequencies of the spring system? (Expect irrational numbers.)



6. A spring with 8Ns/m damping, a 2kg mass, and a 8N/m Hooke coefficient.

   Suppose the spring is initially stretched 1 meter up from equilibrium then released with -1 m/s initial velocity. After 20 seconds, a driving force of $4\sin 2(t-20)$ is applied to the mass. Compute the position of the spring (as measured from the equilibrium) for all positive time.

(b) Compute the gain as a function of the frequency. At what frequency is the gain maximized?

7. Compute the matrix exponential e^{At} if

$$A = \begin{pmatrix} 6 & -15 & 29 \\ 3 & -6 & 6 \\ 0 & 0 & -2 \end{pmatrix}$$

8. An investment account is opened with \$3,000 on Jan 1, 2017. The account pays an annual .8% continuously compounded interest rate. Continuous withdrawals are made at a rate of \$120 per year. A \$2,000 is made on Jan 1 of 2018 and a \$1,000 deposit is made on Jan 1 of 2025. Write an initial value problem to model the situation. What are the quantities described by all of your variables and what units do they carry? Solve for the value of the account for all positive time.

9. Find the general solution to the differential equations:

   a) $y'' + y' - 12y = te^{3t}$

   b) $y = (4-y)(5-y)t$

   c) $ty' + 2y = 4t^2$

   d) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$

   e) $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-4}}$

f) $2x + 3 + 2yy' - 2y' = 0$

g)

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

10. Compute the Laplace transform of the following functions:

a)

$$t \sin(3t)$$

b)

$$e^{6t} u(t-3)$$

c) The periodic function

$$f(t) = \begin{cases} t & \text{if } t \in [0,3) \\ 0 & \text{if } t \in [3,5) \\ f(t-5n) & \text{if } t \in [5n, 5n+5) \text{ for any positive integer n} \end{cases}$$

d)

$$\frac{\delta(t-22)}{t}$$

e)

$$g(t) = \begin{cases} 0 & \text{if } t < 10 \\ (t-10)^3 & \text{if } 10 \leq t < 13 \\ 0 & \text{if } t > 13 \end{cases}$$

11. Compute the inverse Laplace transform of the following frequency functions

a)

$$\frac{e^{-5s}}{s^2 + 25}$$

b)

$$\frac{s}{(s-1)(s-2)(s-3)}$$

c)

$$\frac{1}{1 - e^{-6s}}$$

d)

$$\frac{5s + 25}{s^2 + 10s + 74}$$

12. Suppose two species prey population p and predator population q evolve according to

$$\frac{dp}{dt} = p(-1 + 2.5p - 0.3q - p^2)$$
$$\frac{dq}{dt} = q(-1.5 + p)$$

Find the critical points and determine the stability of each. Determine the possible limiting values of the populations and decide what initial conditions result in those values.

13. Suppose a differential equation in polar coordinates is given by

$$\frac{dr}{dt} = |r - 3|r(r + 2)(r - 5) \qquad \frac{d\theta}{dt} = 3.$$

Determine all the periodic solutions. Determine the limiting behavior of solutions for any initial value. Sketch the solutions in rectilinear space.