Math 3012-L
Spring 2018
Exam 1
1 Feb
Time Limit: 70 Minutes

Name: ____________________

This exam contains 7 pages (including this cover page) and 6 questions. There are 25 points in total. Justify all answers. Any expression for a number is acceptable; there is no need to find a decimal representation. Write explanations or proofs clearly and in complete thoughts. Points are reserved for clarity. Use the blank side of paper for scratch work. No calculators or notes may be used.

On my honor, I pledge that I will not give or receive aid in examinations; I will not use unapproved materials in examinations; I will not misrepresent my work or represent the work of another as my own; and I will avoid any activity which will encourage others to violate their own pledge of honor.

Signature: ____________________

Print Name: Jack

Formal Symbols Crib Sheet

$\neg$	not	$\wedge$	and	$\vee$	or
$\Rightarrow$	implies	$\lightning$	contradiction	$\in$	element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$	equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$	integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv \pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$	complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$	set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$	big-O asymptotic order
2^A	power set of set A	$\lvert A\rvert$	cardinality of set A	A^B	set of functions $B \to A$

1. (6 points) (a) What is an *injective function*?

A function $f: A \to B$ is injective if for every pair $a_1, a_2 \in A$ if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

(b) Give an example of an injective function from the integers $\mathbb{Z}$ to the non-negative integers $\mathbb{Z}_{\geq 0}$.

$g: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$

with $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 2|x|+1 & \text{if } x < 0 \end{cases}$ is injective.

(c) How many injective functions are there from the set $\{0, 1, 2\}$ to the set $\{1, 3, 5, 7, 9\}$?

$5 \cdot 4 \cdot 3$

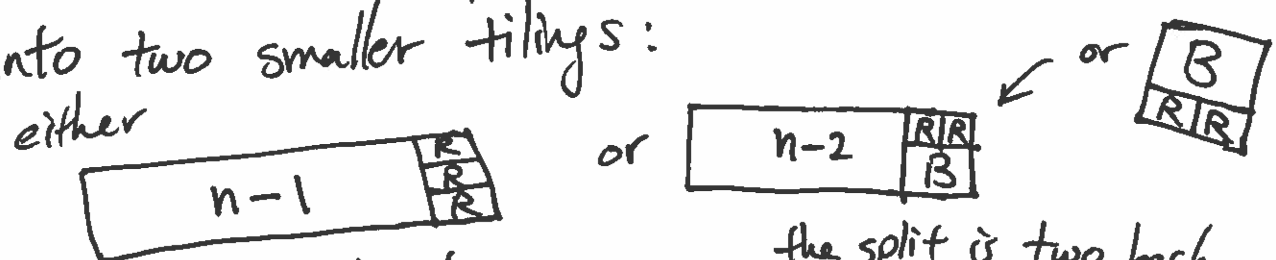
Since each the you pick an image, you lower your choices by 1.

2. (6 points) A $3 \times n$ checkerboard is to be tiled using two types of tiles. The first type of tile is a red 1×1 square tile. The second type of tile is a blue 2×2 square tile. Give (and explain) a recursive formula for the number of possible tilings. Use it to compute the number of tilings of a 3×4 board.

Let a_n be the # of tilings.

$a_0 = 1 \quad a_1 = 1 \quad a_2 = 3$

Find the rightmost ~~place~~ vertical line you could split a board into two smaller tilings:



the split is 1 back and the right is all red

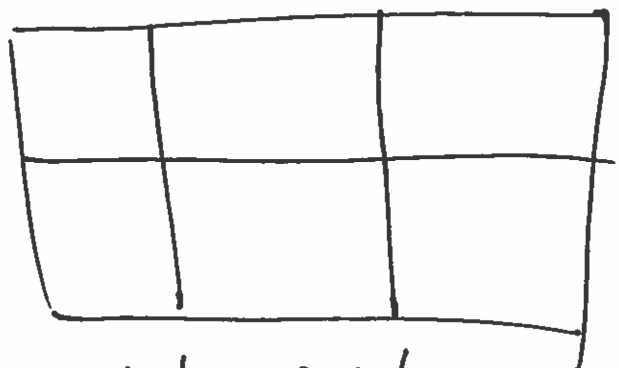
the split is two back with a blue & 2-reds either blue up or down.

So $a_n = a_{n-1} + 2a_{n-2}$

$a_0 = 1 \quad a_1 = 1 \quad a_2 = 3 \quad a_3 = 5 \quad \boxed{a_4 = 11}$

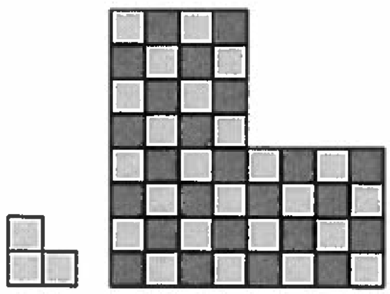
3. (3 points) Prove that if 7 points are chosen anywhere in a 2 cm $\times$ 3 cm rectangle, then there must be a pair of points no more than $\sqrt{2}$ cm apart.

Split the rectangle into 6 $1 \times 1\ cm^2$

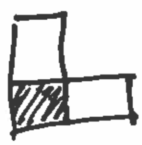
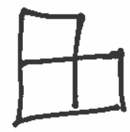


By pigeon-hole-principle some square has a pair of points. Since the diameter of the squares is $\sqrt{2}$ cm, that pair must be within $\sqrt{2}$ cm. ▨

4. (3 points) The L-shaped checkerboard with $3k^2$ squares can be made by cutting the top right $k \times k$ square out of a $2k \times 2k$ square board for any positive integer k. The L-shaped checkerboard with $3 \cdot 4^2$ squares is shown below. L-shaped tiles can be made by cutting the top right square out of a 2×2 square tile, as shown below. Prove that an L-shaped checkerboard with $3 \cdot (2^n)^2$ squares can always be tiled by L-shaped tiles for any integer $n \geq 0$.

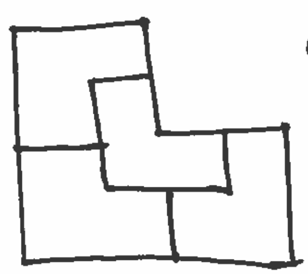


Induct on the size n.

Certainly  is tiled by  so it's true for $n=1$

Suppose that there is $k \in \mathbb{Z}$ with $3 \cdot (2^k)^2$ tiled by .

Then split the $3 \cdot (2^{k+1})^2$ board into four sub-Ls which are size $3 \cdot (2^k)^2$ Ls.

like so: 

By ~~induction~~ inductive assumption they can be tiled, so

the $3 \cdot (2^{k+1})^2$ board can be tiled. ■

5. (8 points) Consider the *quaternary* strings on the alphabet $\{0, 1, 2, 3\}$.

(a) How many length 8 quaternary strings are there?

4^8 since a choice of 4 options is made 8 times.

(b) How many length 10 quaternary strings have exactly 5 ones?

Pick where the ones go : $\binom{10}{5}$ ways

Then Decide $\{0,2,3\}$ for the other 5 positions : 3^5 ways

Total: $\binom{10}{5} \cdot 3^5$

(c) How many length 16 quaternary strings have no digit less than the previous digit?

Bijects to ~~$\mathbb{Z}$~~ solutions of: $X_0 + X_1 + X_2 + X_3 = 16$

Kids & Candies count gives $\binom{16+4-1}{4-1}$ ways

(d) How many length 26 quaternary strings have the property that their digits sum to exactly 3?

Mostly the string is zeros, except: Could have one 3 xor one 2 ~~two~~ 1 xor three 1s

$$\binom{26}{1} + \underset{\substack{\uparrow \\ 12 \text{ or } 21}}{2} \cdot \binom{26}{2} + \binom{26}{3}$$

pick the 3 place

pick 2 spots

pick 3 spots for the ones

6. (5 points) (a) State the Binomial Theorem.

If x, y are polynomial variables then $\forall n \in \mathbb{Z}_{\geq 0}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(b) Recall that a *palindrome* is a string of English letters $\{A, B, \ldots, Z\}$ that reads the same backwards and forwards. The palindrome DO GEESE SEE GOD repeats the letter E 5 times. What's the longest a palindrome could be without repeating a letter 6 times?

If length n the first $\lfloor \frac{n}{2} \rfloor$ letters are repeated again in the last half.

If $\lfloor \frac{n}{2} \rfloor > 2 \cdot 26$ we'd have 3 letter repeats in the first half $\Rightarrow$ 6 repeats in the palindrome. So $\lfloor \frac{n}{2} \rfloor \leq 2 \cdot 26$ implies $n \leq 4 \cdot 26 + 1$ which we can attain with just A repeated 5 times:

$$\overbrace{AB\ldots Z}^{26}\ \overbrace{AB\ldots Z}^{26}\ A\ \overbrace{ZY\ldots A}^{26}\ \overbrace{ZY\ldots A}^{26}$$

(c) BONUS: Recall that an *anagram* is a rearrangement of the positions of letters in a string. What is the maximum number of distinct anagrams that a palindrome of that length could have?

$$\frac{(4 \cdot 26 + 1)!}{5! \cdot 4!^{25}}$$

The string above has anagrams, which is max since any more repeats will increase the denominator.

Count by permuting all letters, the dividing out the permutations of identical letters.