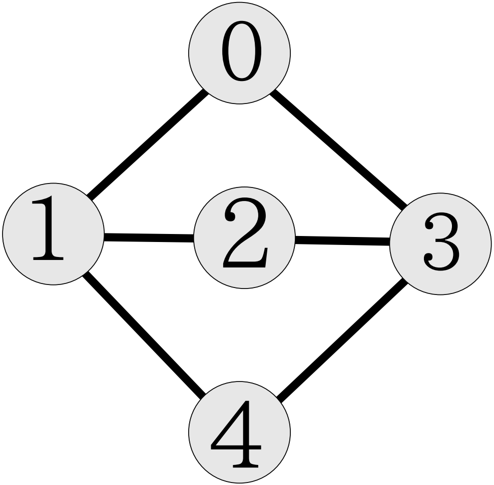
Give the closed form generating function for the following:

1. the number of length n strings of $\{A, C, G, T\}$
2. the number of n digit decimal numbers
3. the number of strings of $\{1, 3, 5, 7\}$ whose digits sum to n
4. the number of ways to make n¢ postage from 21¢ and 34¢ stamps
5. the number of length n strings of $\{0, 1, 2, 3, 4\}$ where a 3 or 4 never follows directly a after 0, 1, or 2

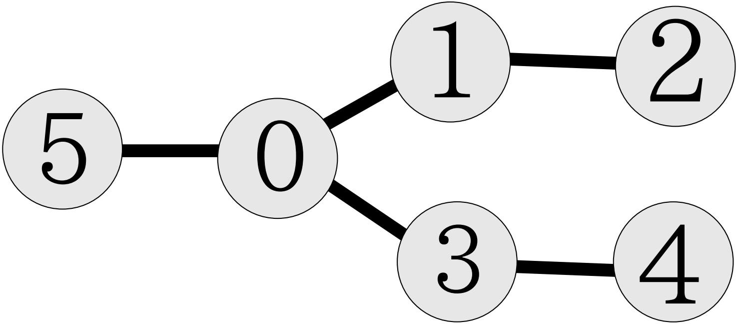
Alice and Bob play a game. Every round they flip a coin. If the coin is heads Alice rolls a 6-sided die and gets that many points. If the coin is tails Bob rolls a 6-sided die and gets that many points. How many ways can the game end with final score Alice 5 to Bob 10. (Use a generating function)

Prove by a combinatorial argument that

$$\left\{ {k \atop n} \right\} = n \left\{ {k-1 \atop n} \right\} + \left\{ {k-1 \atop n-1} \right\}$$



What is the Polya index of the group of isomorphisms of the graph?



How many was can we paint the vertices cyan, yellow, magenta, or black up to isomorphisms of the graph? What is the probability that all colors are represented if all distinct paintings are equally likely?

Solve the recurrence relation $a_{n+2} = 5a_n - a_{n-2}$ where $a_0 = a_1 = 0$ and $a_2 = 1$ and $a_3 = 4$.

A non-negative integer solution to $a + b + c + d + e = 42$ is chosen at random. What's the probability that $a \geq 8$ and $b \leq 3$?

Let C be a Hamiltonian cycle on a graph with vertices labeled $\{1, \ldots, 36\}$. Prove that there is a path with 3 vertices in C whose vertex labels sum to at least 56.

Jim Bund is in a shootout with 10 of Silvertoe's agents. He fires all 13 of his bullets randomly from behind cover. He's sure every bullet hit an agent. What's the probability that all 10 were hit and it's safe to come out? What's the probability that only 8 were hit and he should still cover?

Division Lattice Flow

Consider the directed network whose vertices are all the divisors of 120.

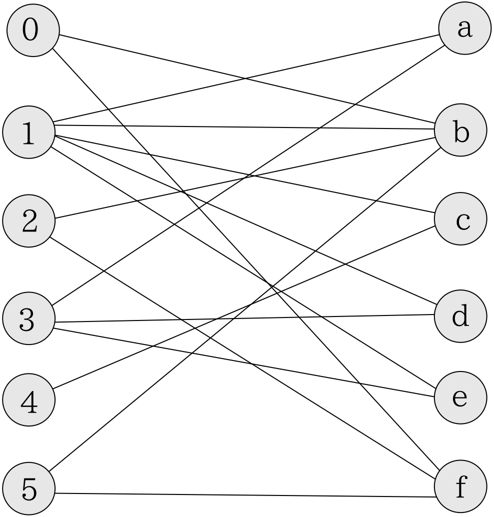
$$V = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$$

with an arc xy of capacity $\frac{y}{x}$ if $\frac{y}{x}$ is a prime number. (This is the *division lattice* of 120). Compute a maximal flow and a minimal cut from source 1 to sink 120.

What is the probability that a rearrangement of the letters ONOMATOPOEIA conatins the subword POEN?

Order by increasing $\mathcal{O}$:

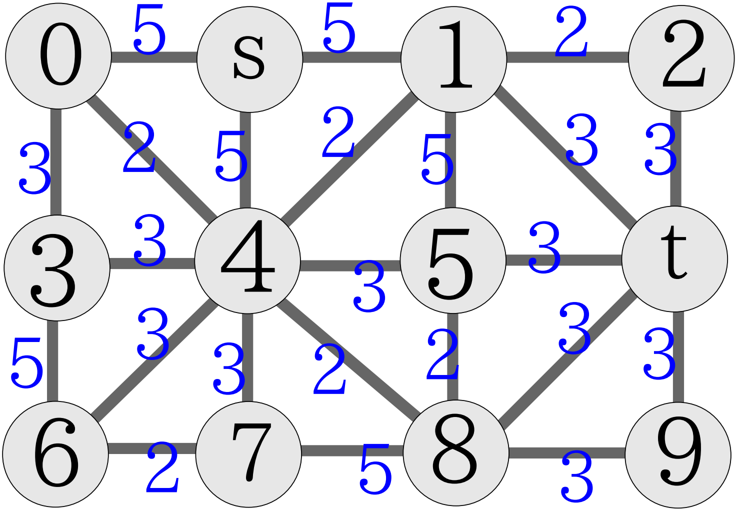
1. $\log((n+1)!)$
2. 5
3. $\left(\frac{1}{2}\right)^n$
4. $5 + \frac{1}{n} + n^3$
5. $(n-1)\log(n)$



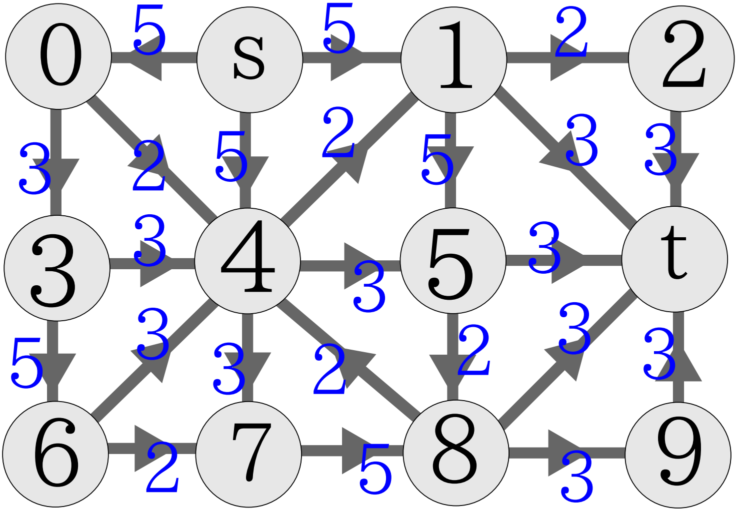
Compute a maximal matching.

Describe an algorithm that can solve the following problems. Is there a known algorithm of complexity **P**, **NP**, or **EXP**? If **NP** describe the certificate.

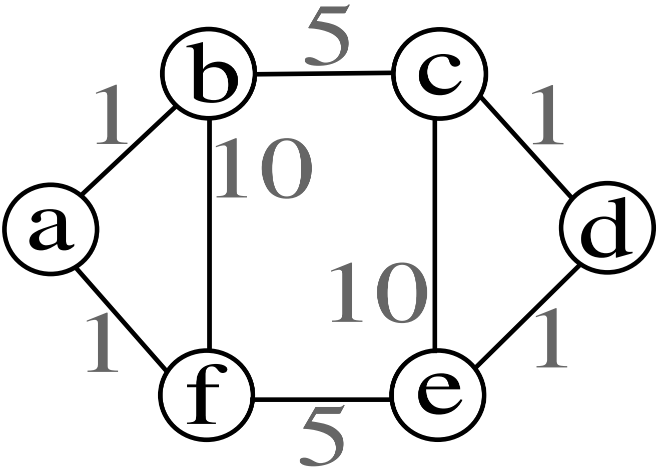
1. Given a size n set of positive integers, can the set be split into two subsets with the same sum?
2. Given a weighted graph with n vertices, is there a spanning tree of weight less than c?
3. Given a weighted graph with n vertices, is the graph 4-colorable?
4. Given a weighted graph with n vertices, is there a walk of weight less than c containing every vertex?
5. Given a weighted graph with n vertices, is there a walk of weight less than c containing every edge?
6. Given a weighted graph with n vertices, what is the minimal weight walk containing every edge?



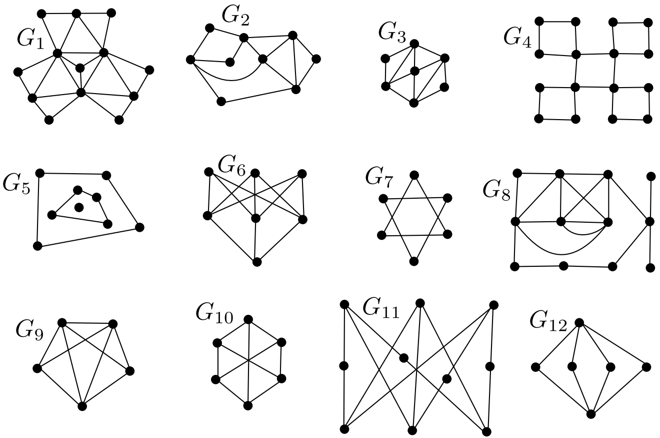
Compute a minimal spanning tree. Compute a minimal weight walk containing every edge. Explain why a minimal weight walk of any weighted graph never covers any edge more than twice.



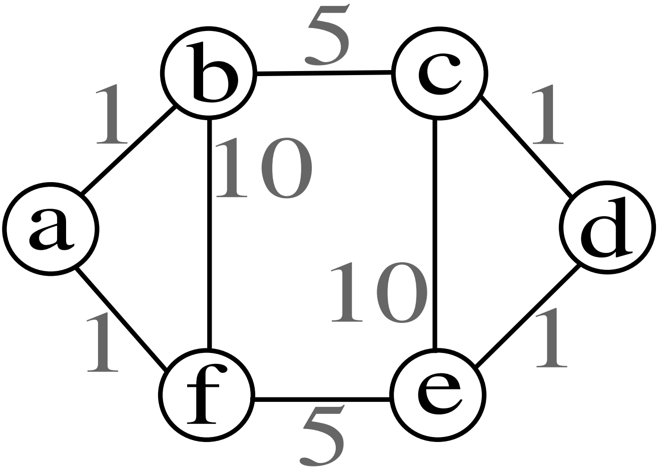
Compute a maximal flow and a minimal cut.



Compute the weight of a minimal weight closed walk containing every edge.



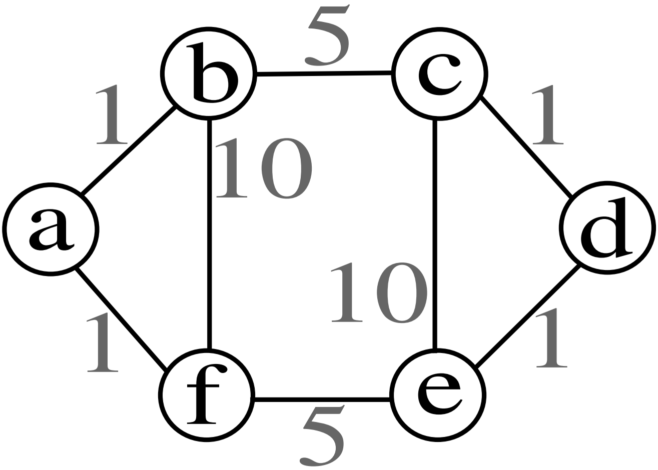
Which graphs are planar? Eulerian? Hamiltonian? 4-colorable?



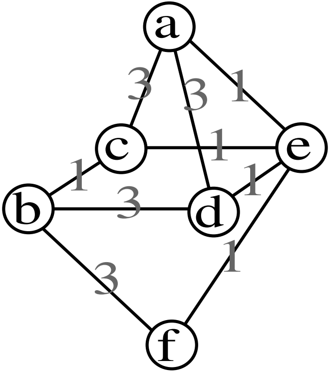
Give a minimal perfect matching.

$N = 6331951901953125 = 3^9 5^8 7^7$

What is the probability that a randomly chosen positive factor of N is the product of exactly 10 (non-distinct) primes? Given that it is divisible by 3^5?



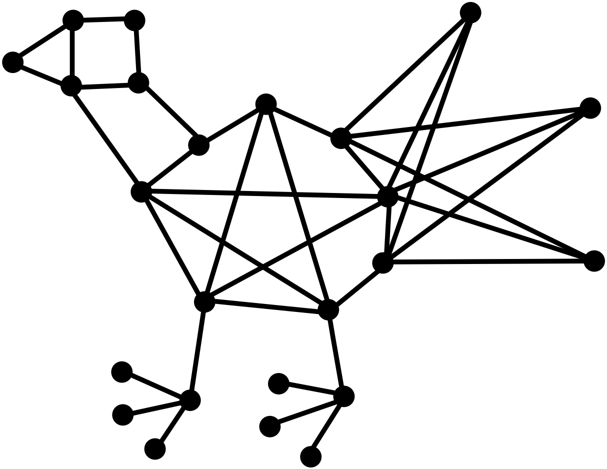
How many subgraphs does the graph have? What is the width of the poset of subgraphs of the graph?



Compute a minimal weight circuit containing every edge. Compute a minimal spanning tree.

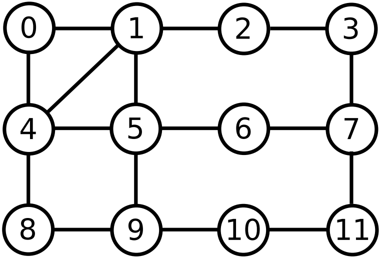
Which of the following are permutation groups?

1. the set of derangements of $\{1, 2, 3, 4, 5, 6\}$
2. the set of isomorphisms of a graph (V, E) to itself that leave vertex $0 \in V$ unchanged
3. the set of bijections $\{a, b, c\} \to \{1, 2, 3\}$

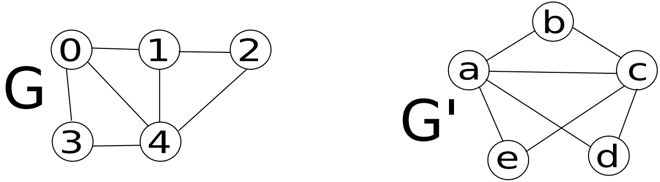


Is the graph planar? Prove it.

Prove by induction on the number of vertices that every tree is 2-colorable.



Is the graph Eulerian? Prove it. Is the graph Hamiltonian? Prove it. What is the chromatic number?



Is the graph G connected? bipartite? complete?
Are the graphs G and G' isomorphic? **Prove** your claim.

How many distinct permutations are there of the phrase FULL MOON MONSTER?

Prove that the chromatic number of a graph is at most one more than the maximal degree among the vertices. (Induct.)

Ponder

Alice has a red, yellow, and blue ball. Can she obtain every combination by adding or deleting one ball at a time and never repeat a combination?

Which complete bipartite graphs are Hamiltonian?

Prove that $K_{n,m}$ is Hamiltonian if and only if $n = m$.

Nice Necklace

Prove by counting necklaces that $a^p \equiv a \pmod{p}$ for any prime number p and positive integer a.

Find the complexity!

INPUT: Input weight w on $K_{n,n}$
OUTPUT: Minimal weight m Hamiltonian cycle M

I. Set $m = \infty$ and M the empty graph
II. For every Hamiltonian cycle H of $K_{n,n}$:
   A. Compute the weight of H:
      i. Set $s = 0$.
      ii. For every edge $e \in H$ add the weight to s. So $s \leftarrow s + w(e)$
   B. If $m > s$ replace m with s and M with H
III. Return m and M

Which functions are one-to-one? onto? bijective

a. the function $f : \{2, \ldots, 10\} \to \{2, \ldots, 10\}$ where $f(x)$ is the smallest prime that divides x

b. a function $g : \{0, \ldots, 9\} \to \{0, \ldots, 9\}$ where $g(x) \equiv 3x \pmod{10}$

c. a function $h : \{0, \ldots, 4\} \to \{0, \ldots, 4\}$ where $h(x) \equiv 4x^{25} \pmod{5}$

d. the function sending subgraphs of K_5 to the number of edges in the subgraph $\{0, \ldots, 10\}$

Prove that $\forall n \geq 1$:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

Blangon particles create a time dependent chain reaction. In the first second a blangon exists it will spawn 4 new blangons. In the second second a blagon will spawn 8 new blangons. Thereafter the blagon will destroy 12 of the newest blangons each second. Whats the long term growth rate of a blagons chain reaction?

Random Relation

A binary relation is chosen at random on $\{0, 1, 2, 3, 4\}$. What is the probability that it is reflexive? What is the probability that it is symmetric? What is the probability that it is antisymmetric? Are these independent events?

Which relations are REFLEXIVE?

1. *within distance 6* on any graph
2. *coprime* on $\mathbb{Z}_+$
3. *subgraph* on a set of graphs
4. *homeomorphic* on a set of graphs
5. $= \mathcal{O}$ on sets of functions

Which relations are SYMMETRIC?

1. *within distance 6* on any graph
2. *coprime* on $\mathbb{Z}_+$
3. *subgraph* on a set of graphs
4. *homeomorphic* on a set of graphs
5. $= \mathcal{O}$ on sets of functions

Which relations are ANTISYMMETRIC?

1. *within distance 6* on any graph
2. *coprime* on $\mathbb{Z}_+$
3. *subgraph* on a set of graphs
4. *homeomorphic* on a set of graphs
5. $= \mathcal{O}$ on sets of functions

Which relations are TRANSITIVE?

1. *within distance 6* on any graph
2. *coprime* on $\mathbb{Z}_+$
3. *subgraph* on a set of graphs
4. *homeomorphic* on a set of graphs
5. *independence* on the events of a 10 coin flip experiment (sample space $X = \{H, T\}^{10}$)

Which relations are partial orders?

1. *within distance 6* on any graph
2. *coprime* on $\mathbb{Z}_+$
3. *subgraph* on a set of graphs
4. *homeomorphic* on a set of graphs
5. $= \mathcal{O}$ on sets of functions