Math 3012-A
Summer 2015
Exam 1
10 June 2015
Time Limit: 70 Minutes

Name: ______________________

This exam contains 8 pages (including this cover page) and 8 questions. There are 42 points in total. Any expression for a number is acceptable; there is no need to find a decimal representation. Write explanations or proofs clearly and in complete thoughts. Points are reserved for clarity. Do not include any scratch work that is not part of the proof in the proof. Use the blank side of paper for scratchwork. No calculators or notes may be used. Put your name on every page.

Grade Table

Question	Points	Score
1	6	
2	2	
3	2	
4	4	
5	10	
6	6	
7	4	
8	8	
Total:	42	

Formal Symbols Crib Sheet

$\neg$	not	$\wedge$	and	$\vee$	or
$\Rightarrow$	implies	$\lightning$	contradiction	$\in$	element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$	equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$	integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv \pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$	complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$	set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$	big-O asymptotic order
2^A	power set of set A	$\lvert A\rvert$	cardinality of set A	A^B	set of functions $B \to A$

1. (6 points) (a) Give the definition of a *bijection*.

(b) Give a bijection between the set of portraits below labeled by 9 and the set

$N = \{Alito, Breyer, Ginsburg, Kagan, Kennedy, Roberts, Scalia, Sotomayor, Thomas\}$

Bonus if your bijection is *name*.

(c) How many bijections are there between N and 9?

2. (2 points) Female honey bees have a mother and father, but male bees have only a mother. (E.g. a male bee has 1 parent, 2 grandparents, 3 great-grandparents, ...) Write a recurrence relation for the number of ancestors a bee has n generations back.

3. (2 points) What's wrong with this proof?

> <u>Everyone is the same age.</u>
>
> Proof: Induct on the number of people. Everyone in a set of 1 person is the same age, so the base case of one person holds. Suppose that any k people are the same age. Let $p_1, \ldots, p_k, p_{k+1}$ be $k+1$ people. Then $p_1, \ldots, p_k$ form a set of k people, and so all the same age by the inductive hypothesis. Similarly $p_2, \ldots, p_k, p_{k+1}$ are all the same age. But p_2 is in both sets so all $k+1$ people are the same age. By induction any finite number of people are the same age. □

4. (4 points) Ten points are chosen in an equilateral triangle of side length 1. Prove there must be a pair of points that are no more than distance $\frac{1}{3}$ apart.

5. (10 points) The set of (uncased) English letters is $\{A, B, C, \ldots, Z\}$ and has 26 elements.
   (a) How many length 10 strings of English letters are there?

   (b) How many distinct strings can one formed by rearranging the letters of SASSAFRASTASTE?

   (c) How many strings of English letters never repeat a letter and are in alphabetical order?

   (d) How many length 40 strings of English letters are in alphabetical order and may have repeating letters?

   (e) How many length 40 strings of English letters are in alphabetical order and have every letter appear at least once?

   (f) Bonus: How many people are required to guarantee that there is a set of at least 10 people whose first names begin with the same letter and whose last names begin with the same letter?

6. (6 points) *Partition Sum Problem:* Given a length n string of integers a, is there a partition the terms of a into two disjoint subsets with the same sum? i.e., are there sets I, J with $n = I \cup J$ with $I \cap J = \varnothing$ and $\sum_{i \in I} a_i = \sum_{j \in J} a_j$?

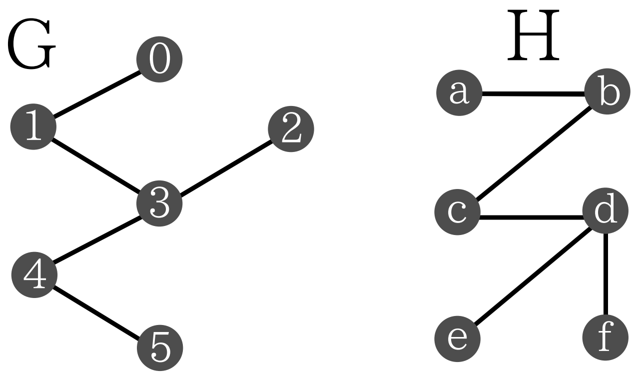
   (a) Describe an algorithm that can solve the partition sum problem.

   (b) Estimate the $\mathcal{O}$ complexity of your algorithm if the basic operation is summing two numbers.

   (c) Based on your algorithm above...
   True / False / Cannot determine: The Partition Sum Problem is in the class $\mathcal{P}$.

   (d) True / False / Cannot determine: The Partition Sum Problem is in the class $\mathcal{NP}$.

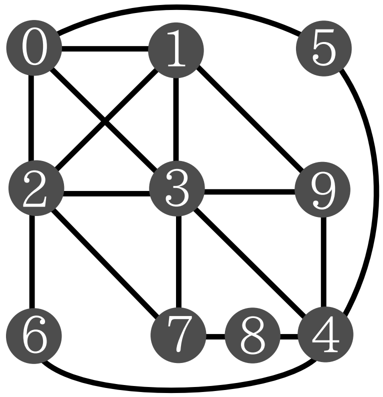
7. (4 points) Consider the connected graphs G and H shown below.



(a) Are G and H isomorphic? Give an isomorphism or explain why one cannot exist.

(b) Recall that the *symmetries* of G are the isomorphisms $G \to G$. How many symmetries does G have?

8. (8 points) Consider the graph shown below.



(a) Is the graph Eulerian? Explain how you know.

(b) Is the graph Hamiltonian? Explain how you know.

(c) Is the graph planar? Explain how you know.

(d) What is the chromatic number of the graph? Explain how you know.