Math 3012-A **Name:** _____
Summer 2015
Final
30 July 2015
Time Limit: 70 Minutes

This exam contains 12 pages (including this cover page) and 10 questions. Only answer 9 questions. Cross off the question you do not want graded from the grade table below. If you do not cross off a question the first 9 will be graded. There are 54 points in total. Put your name on every page. No calculators or notes may be used. Any expression for a number is acceptable. You may freely reference any Twelvefold way number or coefficient of a generating function. There is no need to find a decimal representation.

Honor Pledge: I have read and understand the exam instructions. I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

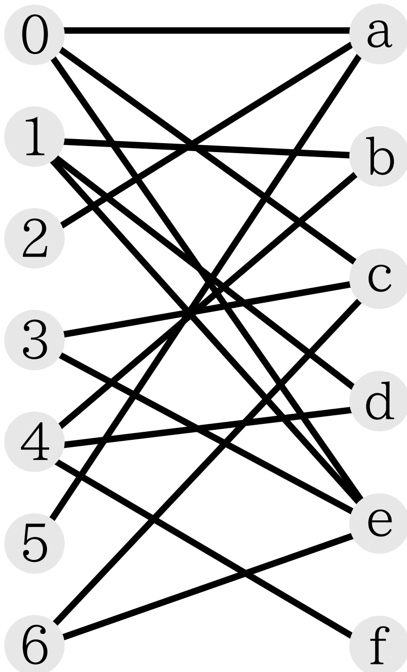
Signature: _____

Grade Table

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	6	6	6	6	6	6	6	6	6	54
Score:											

Formal Symbols Crib Sheet

$\Rightarrow$	implies	$\lightning$	contradiction	$\in$	element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$	equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$	integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv \pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$	complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$	set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$	big-O asymptotic order
2^A	power set of set A	$\lvert A \rvert$	cardinality of set A	A^B	set of functions $B \to A$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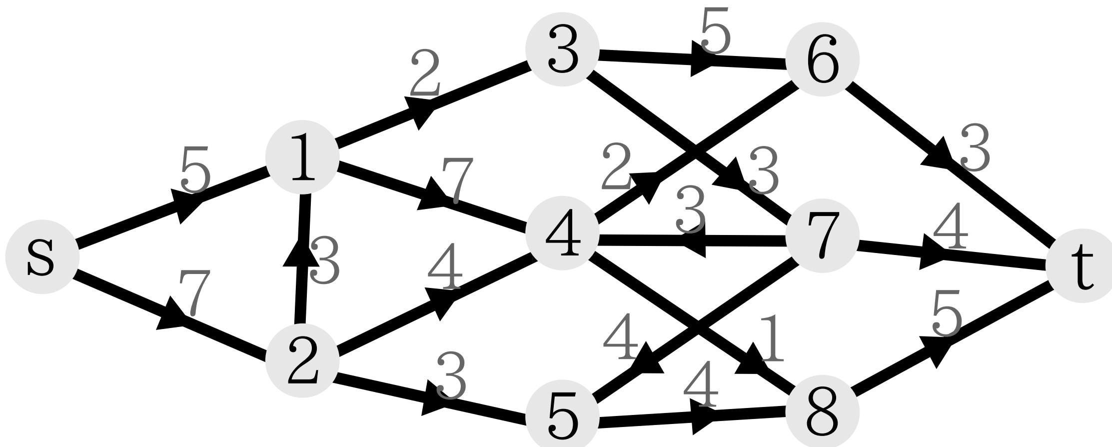
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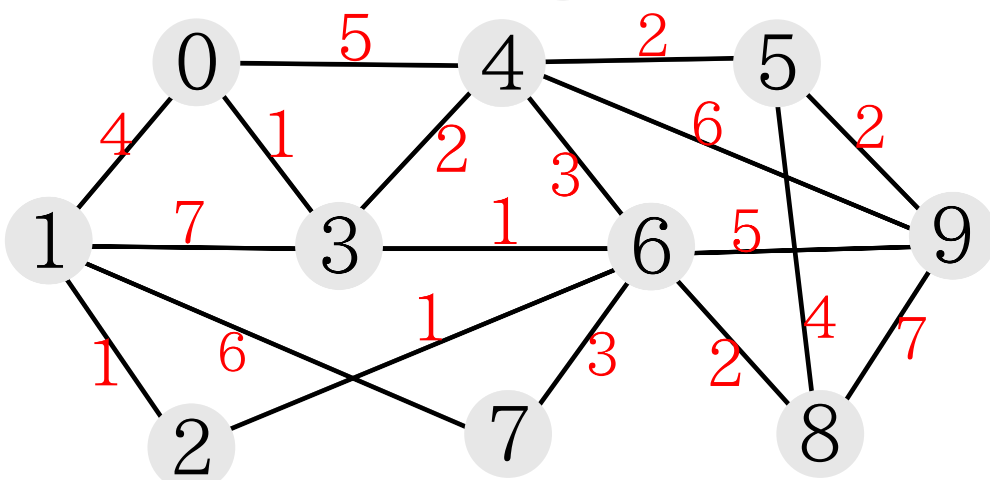
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1. (a) (3 points) How many distinct ways can one rearrange the letters of the string MUSTMASTERMATH?

   (b) (3 points) Give the closed form of the generating function for the number of strings of $\{1, 2, 4\}$ whose digits sum to n.

2. (a) (4 points) Match each number in Set 1 with an equal quantity in Set 2.

Set 1

A. the number of length 25 decimal strings

B. the number of length 25 binary strings with 10 ones

C. the number of length 25 decimal strings containing every decimal digit

D. the number of length 25 decimal strings where the digits appear in non-increasing order

E. the number of decimal strings where the digits appear in non-increasing order and the length plus digits sum to 15

Set 2

I. $\binom{25}{10}$

II. $\binom{25+10-1}{25}$

III. $p_{10}(25)$ the number of integer partitions of 25 into 10 or fewer integers

IV. $s(25, 10)$ the number of surjections from 25 to 10

V. 10^{25}

(b) (2 points) If you answer the above question by guessing a random matching, what is the probability of not getting any correct at all?

3. (6 points) Prove one of the following two statements. Circle the statement you are proving.

   1. Let C be a Hamiltonian cycle on a graph with vertices set $\{1, 2, \ldots, 10\}$. Prove that there is a path of 5 vertices in C whose vertex labels sum to at least 28.
   2. Prove by induction that for every integer $n \geq 2$ we have $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$.

4. For any non-negative integer n let a_n be the number of strings of $\{0, 1, 2, 3, 4, 5, 6\}$ that never have a 0, 1, or 2 appear after a 3, 4, 5, or 6.

   (a) (1 point) Give a linear recurrence relation satisfied by a_n.

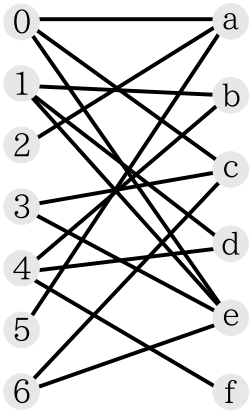
   (b) (2 points) Give a closed form generating function for a_n.

   (c) (3 points) Give a formula for a_n in terms of n.

5. (a) (3 points) What distinguishes problems in complexity class **NP** from other **EXP** problems?

(b) (3 points) Milk is sold in jugs with volume $\frac{1}{4}$ gallon, $\frac{1}{2}$ gallon, or 1 gallon. How many distinct collections of milk jugs have 42 gallons in total?

6. Consider the graph shown below. (For your convenience there is another copy on the back of the cover page.)

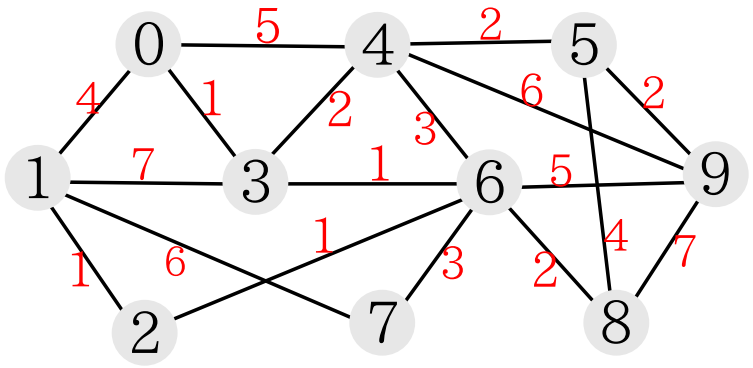


(a) (2 points) What is the chromatic number of the graph? Explain.

(b) (2 points) Is the graph planar? Explain.

(c) (2 points) Compute a maximal matching of the graph.

7. Consider the weighted graph shown below. (For your convenience there is another copy on the back of the cover page.)



(a) (1 point) Is the graph Hamiltonian?

(b) (1 point) Is the graph Eulerian?

(c) (2 points) Compute a minimal spanning tree.

(d) (2 points) Compute the weight of a minimal closed walk containing every edge.

8. (a) (3 points) For the binary relation R and set X below, is R a partial order on X? If no, which property does the relation R lack?

   (a) X is the set of subsets of $\{0, 1, 2, 3, 4\}$. For $x, y \in X$, xRy if there is a surjective function $x \to y$

   A. R is a partial order

   B. R is not a partial order because it lacks the _____ property

   (b) X is the set of humans who have lived on Earth. For $x, y \in X$, xRy if x is y or x is an ancestor of y.

   A. R is a partial order

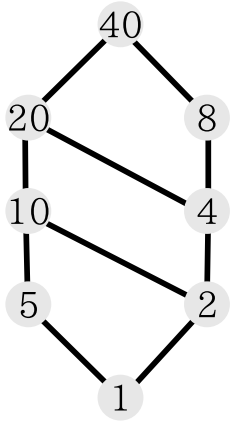
   B. R is not a partial order because it lacks the _____ property

   (c) X is the set of subgraphs of K_7. For $x, y \in X$, xRy if x is homeomorphic to y.

   A. R is a partial order

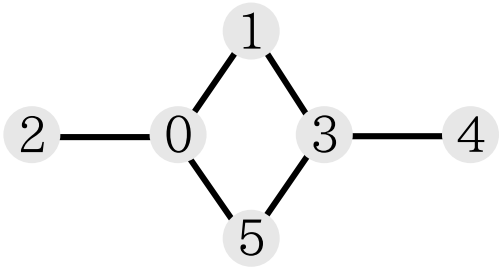
   B. R is not a partial order because it lacks the _____ property

   (b) (3 points) Consider the Hasse diagram for the division lattice of 40, shown below. What is the width? Give a maximal antichain. Give a minimal partition into chains.



9. (a) (2 points) What is the Polya cycle index of the permutation (0)(12)(354)(67)(8)?

(b) (4 points) Consider the graph shown below. How many distinct ways can the vertices be painted red, orange, yellow, green, or blue up to isomorphisms of the graph?



10. (a) (2 points) State the Min-Cut-Max-Flow Theorem.

(b) (4 points) Consider the flow network shown below. (For your convenience there is another copy on the back of the cover page.) Compute a minimal cut. What is the capacity of your cut?