Math 3012-A
Summer 2015
Final
30 July 2015
Time Limit: 70 Minutes

Name: Solution

This exam contains 12 pages (including this cover page) and 10 questions. Only answer 9 questions. Cross off the question you do not want graded from the grade table below. If you do not cross off a question the first 9 will be graded. There are 54 points in total. Put your name on every page. No calculators or notes may be used. Any expression for a number is acceptable. You may freely reference any Twelvefold way number or coefficient of a generating function. There is no need to find a decimal representation.

Honor Pledge: I have read and understand the exam instructions. I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

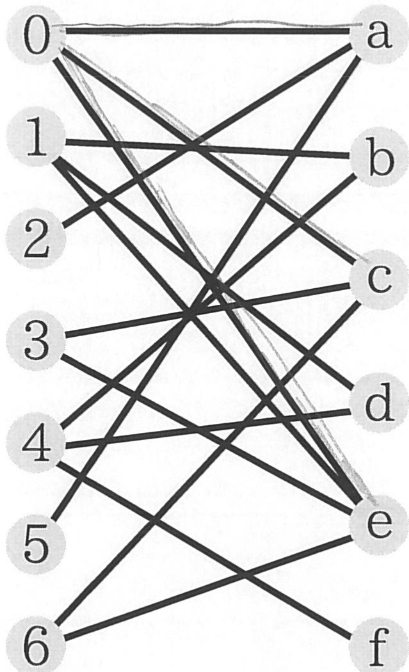
Signature: ______________________________

Grade Table

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	6	6	6	6	6	6	6	6	6	54
Score:											

Formal Symbols Crib Sheet

$\Rightarrow$	implies	$\lightning$	contradiction	$\in$		element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$		equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$		integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv$	(mod n)	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$		complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$		set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$		big-O asymptotic order
2^A	power set of set A	$\|A\|$	cardinality of set A	A^B		set of functions $B \to A$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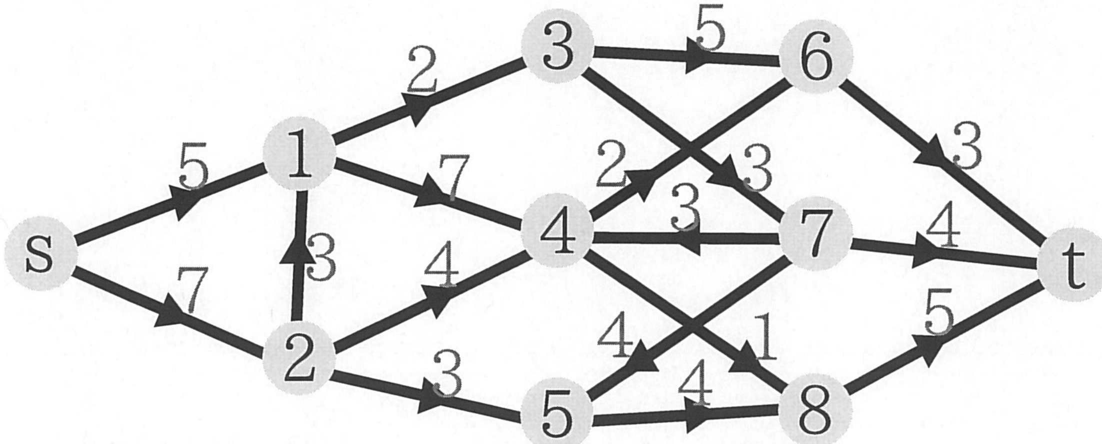
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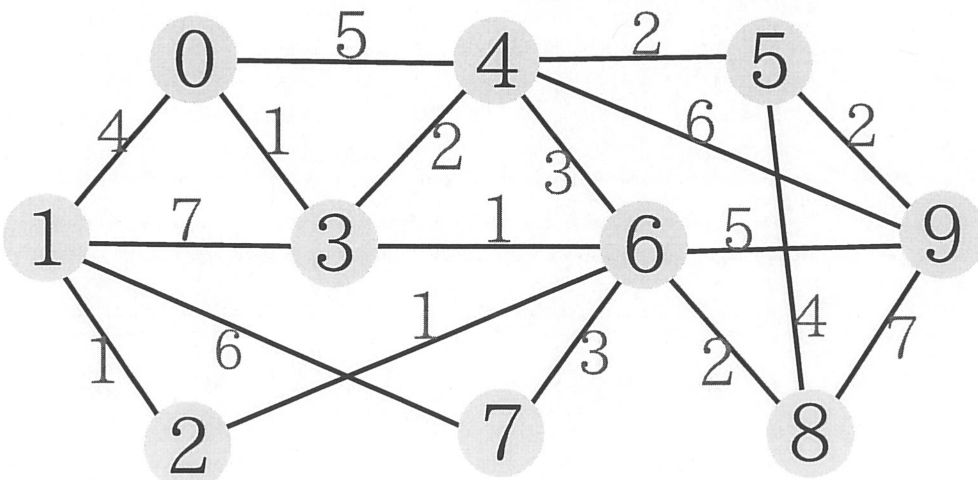
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1. (a) (3 points) How many distinct ways can one rearrange the letters of the string MUSTMASTERMATH?

1 2 3 4 5 6 7 8 9 10 11 12 13 14 3M 1U 2S 3T 2A 1E 1R 1H

Permute the 14 positions in 14! ways then divide out the permutations of Ms, Ss, Ts, and As.

$$\frac{14!}{3!\,2!\,3!\,2!}$$

(b) (3 points) Give the closed form of the generating function for the number of strings of $\{1, 2, 4\}$ whose digits sum to n.

Each digit adds either 1 or 2 or 4 to the sum.

$$D = x + x^2 + x^4$$

So the k-digit strings would have #s of strings summing to n given by generating function

$$D^k = (x+x^2+x^4)^k$$

Since any length string is allowed we sum these disjoint possibilities:

$$\sum_{k\geq 0} D^k = \frac{1}{1-D} = \boxed{\frac{1}{1-(x+x^2+x^4)}}$$

2. (a) (4 points) Match each number in Set 1 with an equal quantity in Set 2.

Set 1

A. the number of length 25 decimal strings

B. the number of length 25 binary strings with 10 ones

C. the number of length 25 decimal strings containing every decimal digit

D. the number of length 25 decimal strings where the digits appear in non-increasing order

E. the number of decimal strings where the digits appear in non-increasing order and the length plus digits sum to 15

Set 2

I. $\binom{25}{10}$

II. $\binom{25+10-1}{25}$

III. $p_{10}(25)$ the number of integer partitions of 25 into 10 or fewer integers

IV. $s(25, 10)$ the number of surjections from 25 to 10

V. 10^{25}

AV DII

BI EV

CIV

(b) (2 points) If you answer the above question by guessing a random matching, what is the probability of not getting any correct at all?

There are 5! ways to biject them and getting all of them wrong would biject to a derangement

so

$$\frac{\text{round}\left(\frac{5!}{e}\right)}{5!} = \sum_{k=0}^{5} \frac{(-1)}{k!}$$

About $\frac{1}{e} \sim 37\%$

3. (6 points) Prove one of the following two statements. Circle the statement you are proving.

   1. Let C be a Hamiltonian cycle on a graph with vertices set $\{1, 2, \ldots, 10\}$. Prove that there is a path of 5 vertices in C whose vertex labels sum to at least 28.
   2. Prove by induction that for every integer $n \geq 2$ we have $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$.

(1) There are 10 paths (undirected) with 5 vertices in C. Replace labels with that many pigeons. If we add the #s of pigeons per path we count each pigeon 5 times. So among the 10 paths there are $5 \cdot \sum_{k=1}^{10} k = 5 \cdot \binom{11}{2} = 5 \frac{11 \cdot 10}{2} = 25 \cdot 11$ $= 275$ pigeons (counting copies). By the pigeonhole principle there is some path with at least $\left\lceil \frac{275}{10} \right\rceil = 28$ pigeons. That path has vertices summing to at least 28.

(2) Induct on the value of n.

Base Case: Since $2 > 1$ we have $\sqrt{2} > 1$ so $\sqrt{2} + 1 > 2$ and $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$.

(Inductive Step) Suppose that there is some n such that $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$.

Then $\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} k + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}}$

by the inductive hypothesis. Then

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n} \cdot \sqrt{n+1} + 1}{\sqrt{n+1}} > \frac{\sqrt{n} \cdot \sqrt{n} + 1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}.$$

So $\sum_{k=1}^{n+1} \frac{1}{k} > \sqrt{n+1}$. This completes the induction.

4. For any non-negative integer n let a_n be the number of strings of $\{0, 1, 2, 3, 4, 5, 6\}$ that never have a 0, 1, or 2 appear after a 3, 4, 5, or 6.

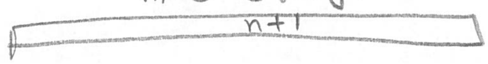
$a_0 = 1$ ← empty string

$a_1 = 7$ ← any length 1 string is fine.

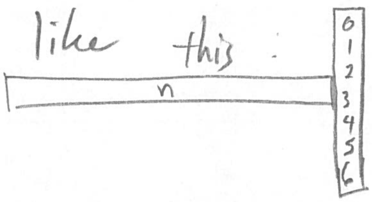
(a) (1 point) Give a linear recurrence relation satisfied by a_n.

$$a_{n+1} = 7a_n - 3\cdot 4\, a_{n-1}$$

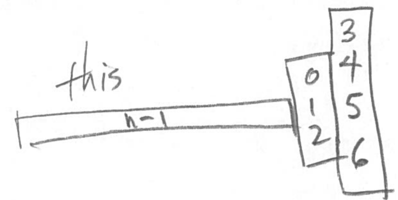
since nice strings



look like this:



but not this



(b) (2 points) Give a closed form generating function for a_n.

$a_{n+2} - 7a_{n+1} + 12a_n = 0$ so if $f(x) = \sum_n a_n x^n$

then

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
-7x f(x) &= \quad -7a_0 x - 7a_1 x^2 - 7a_2 x^3 \\
+12x^2 f(x) &= \quad 12a_0 x^2 + 12a_1 x^2
\end{aligned}$$

$$(1 - 7x + 12x^2) f(x) = a_0 + (a_1 - 7a_0)x = 1 + (7-7)x = 1$$

So $\boxed{f(x) = \dfrac{1}{1 - 7x + 12x^2}}$

(c) (3 points) Give a formula for a_n in terms of n.

Using generating function:

$$\begin{aligned}
f(x) &= \frac{1}{1-7x+12x^2} = \frac{1}{(1-3x)(1-4x)} \\
&= \frac{\frac{1}{(1-4\cdot\frac{1}{3})}}{1-3x} + \frac{\frac{1}{(1-3\cdot\frac{1}{4})}}{1-4x} \\
&= \frac{-3}{1-3x} + \frac{4}{1-4x} \\
&= -3\cdot\sum 3^n x^n + 4\cdot \sum 4^n \cdot x^n \\
&= \sum \left(4^{n+1} - 3^{n+1}\right) x^n
\end{aligned}$$

So $\boxed{a_n = 4^{n+1} - 3^{n+1}}$

OR Linearity:

Char poly $\lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$

so $a_n = c_0 3^n + c_1 4^n$

then $\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

rref $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 7 \end{vmatrix}$ = rref $\begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{vmatrix}$

$\boxed{a_n = 4^{n+1} - 3^{n+1}}$

5. (a) (3 points) What distinguishes problems in complexity class **NP** from other **EXP** problems?

NP problems have a CERTIFICATION

ie, an algorithm to check proposed solutions with

complexity $\mathcal{O}(n^k)$.

(Actually it is unknown if $NP \stackrel{?}{=} EXP$, so potentially nothing distinguishes them!)

(b) (3 points) Milk is sold in jugs with volume $\frac{1}{4}$ gallon, $\frac{1}{2}$ gallon, or 1 gallon. How many distinct collections of milk jugs have 42 gallons in total?

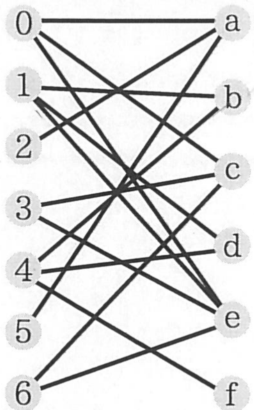
42 is $4 \cdot 42 = 168$ quarter gallons.

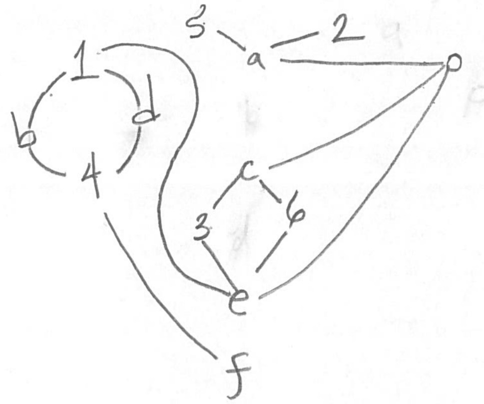
Find the coefficient of

x^{168} in $\dfrac{1}{(1-x)(1-x^2)(1-x^4)}$

which is the generating function for the # of multisets of 1s, 2s, and 4s summing to n.

6. Consider the graph shown below. (For your convenience there is another copy on the back of the cover page.)



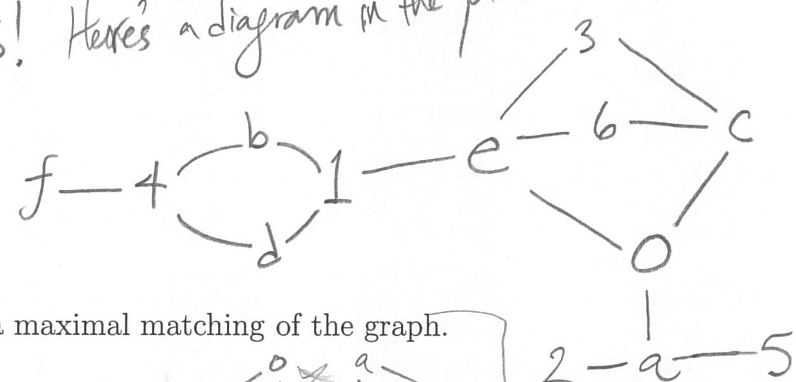


(a) (2 points) What is the chromatic number of the graph? Explain.

2! It's bipartite so you can color the sides.

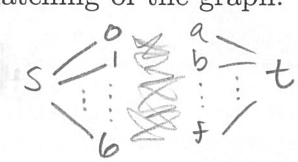
(b) (2 points) Is the graph planar? Explain.

YES! Here's a diagram in the plane:

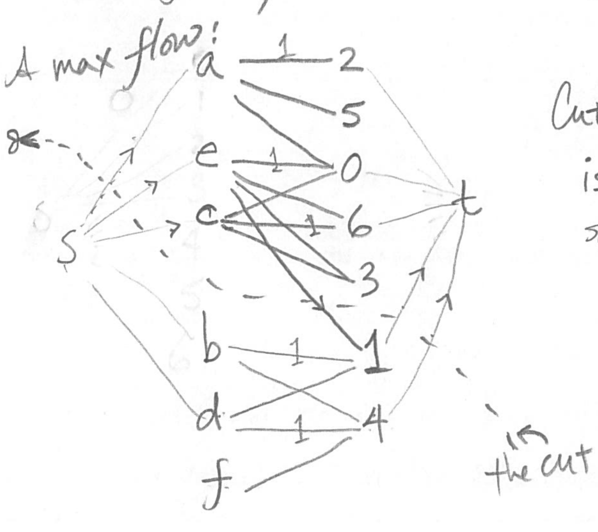


(c) (2 points) Compute a maximal matching of the graph.

Use Ford Fulkerson on



A max flow!

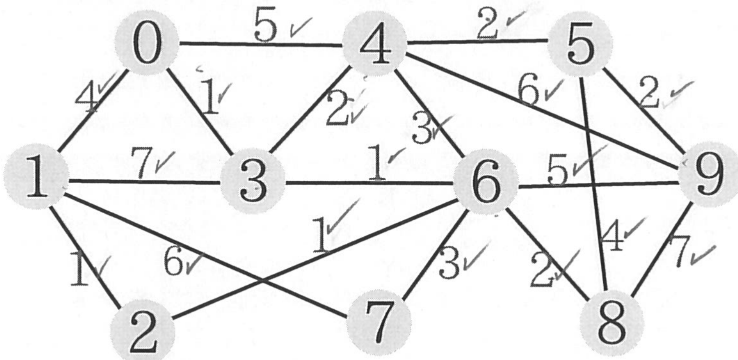


Cut {s, f, 4, d, 1 b} {aec 25063t}

is capacity 5 so a max matching is size 5 and

{a2, e0, c6, b1, d4} is a maximal matching.

7. Consider the weighted graph shown below. (For your convenience there is another copy on the back of the cover page.)



$1\,1\,1\,1\,2\,2\,2\,2\,3\,3\,4\,4\,5\,5\,6\,6$

$= 6 + 2 \cdot \sum_{k=1}^{6} k = 6 + 2 \cdot \binom{7}{2}$

$= 6 + 7 \cdot 6 = 8 \cdot 6 = 48$

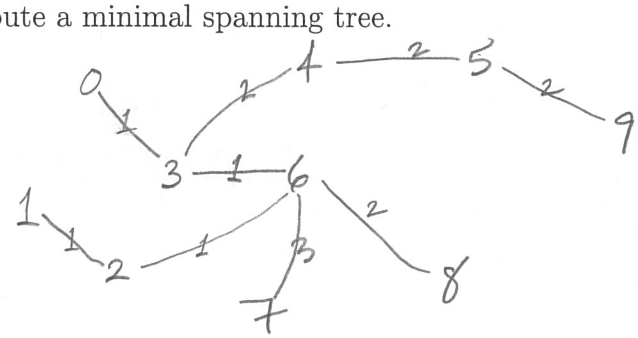
(a) (1 point) Is the graph Hamiltonian?

No way. Vertex 2 and 7 are degree 2 so their incident edges form a 4 cycle, which can't be in a bigger cycle.

(b) (1 point) Is the graph Eulerian?

No. There are odd degree vertices.

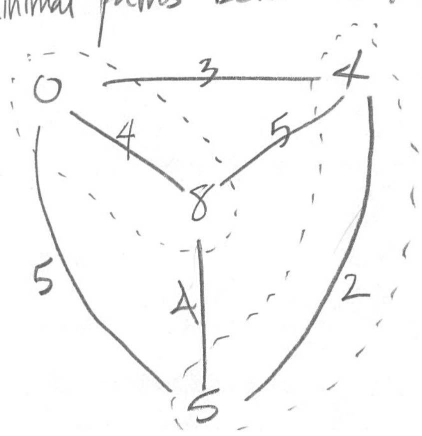
(c) (2 points) Compute a minimal spanning tree.



(d) (2 points) Compute the weight of a minimal closed walk containing every edge.

The odd degree vertices are 0, 4, 5, 8

Find minimal paths between them; the minimal weight perfect match is (08) and (45). Add copies of the edges 03, 36, 68, and 45. to get an Eulerian multigraph. The total weight is



weight $G + 6 = 48 + 6 = 54$

8. (a) (3 points) For the binary relation R and set X below, is R a partial order on X? If no, which property does the relation R lack?

(a) X is the set of subsets of $\{0, 1, 2, 3, 4\}$. For $x, y \in X$, xRy if there is a surjective function $x \to y$

A. R is a partial order it's basically $xRy \simeq |x| \geq |y|$

B. R is not a partial order because it lacks the ______________ property

(b) X is the set of humans who have lived on Earth. For $x, y \in X$, xRy if x is y or x is an ancestor of y. Set is a bit fishy, but the relation is reflexive, transitive, and antisymmetric if you count as your ancestor.

A. R is a partial order

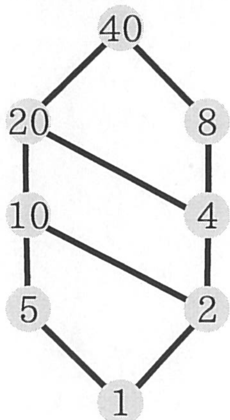
B. R is not a partial order because it lacks the ______________ property

(c) X is the set of subgraphs of K_7. For $x, y \in X$, xRy if x is homeomorphic to y.

A. R is a partial order it's a symmetric relation.

B. R is not a partial order because it lacks the antisymmetry property

(b) (3 points) Consider the Hasse diagram for the division lattice of 40, shown below. What is the width? Give a maximal antichain. Give a minimal partition into chains.



$\{8, 20\}$ is a maximal antichain.

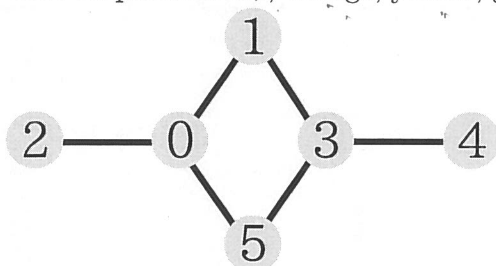
$\{\{2,4,8\}, \{1,5,10,20,40\}\}$ is a partition into 2-Chainz.

9. (a) (2 points) What is the Polya cycle index of the permutation (0)(12)(354)(67)(8)?

$$x_1 x_2 x_3 x_2 x_1$$

$$= x_1^2 x_2^2 x_3$$

(b) (4 points) Consider the graph shown below. How many distinct ways can the vertices be painted red, orange, yellow, green, or blue up to isomorphisms of the graph?



Symmetry group

id

(15)

(03)(24)

(03)(15)(24)

Polya Index of the group is

$$\frac{x_1^6 + x_1^4 x_2 + x_1^2 x_2^2 + x_2^3}{4}$$

By Burnside there are

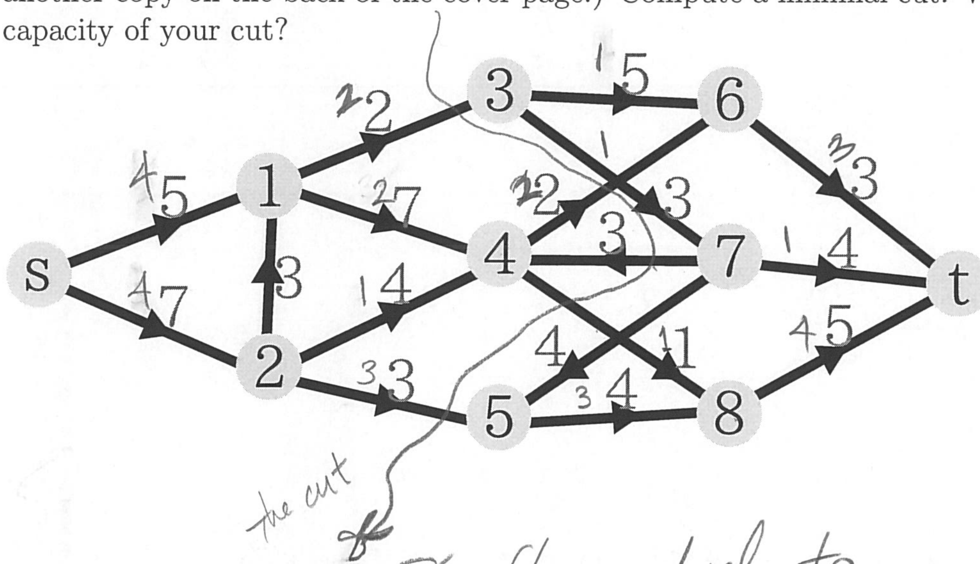
$$\frac{5^6 + 5^5 + 5^4 + 5^3}{4}$$

functions $V \to 5$ up to precomposition with a symmetry.

10. (a) (2 points) State the Min-Cut-Max-Flow Theorem.

The maximum volume amongst all flows on N is equal to the minimum capacity amongst all cuts of N.

(b) (4 points) Consider the flow network shown below. (For your convenience there is another copy on the back of the cover page.) Compute a minimal cut. What is the capacity of your cut?



There is a volume 8 flow dual to the capacity $8 = 2+2+1+3$ minimal cut

$V = \{s, 1, 2, 4\} \cup \{3, 5, 6, 7, 8, t\}$