Computing the Partial Word Avoidability Indices of Ternary Patterns

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Definition

Definition - *pattern, pattern variables, pattern constants*

Let A and E be alphabets with $A \cap E = \varnothing$. We call the letters of E *pattern variables* and denote them by $\alpha, \beta, \gamma, \ldots$ A *pattern* is a word over the alphabet $A \cap E$. A factor $x \in A^+$ of p is called a *pattern constant*.

Examples: square $\alpha\alpha$
overlap $a\alpha a\alpha a$
$\alpha\beta\beta\alpha$
$\alpha\alpha\beta\alpha\alpha\gamma\alpha\gamma\gamma\beta\alpha\alpha\gamma\alpha$

Definition

Definition - *meets, occurs, avoids*

For a word $w \in A^*$ and pattern $p \in (A \cup E)^*$ we say that w *meets* p or p *occurs in* w if there exists some non-erasing morphism $h : (A \cup E)^* \to A^*$ which acts as the identity over A such that $h(p)$ is compatible with a factor of w. We say w *avoids* p when it does not meet p

Examples: $abab$ meets $\alpha\alpha$
$acbcaba$ avoids $a\alpha a\alpha a$
$ababaabc\diamond a\diamond cd\diamond\diamond aba$ meets $\alpha\beta\beta\alpha$

Definition

Definition - *avoidable, unavoidable, k-avoidable, k-unavoidable*

A pattern p is called *k-avoidable* if for a k letter alphabet and for any $n \in \mathbb{N}$ there is a word with n holes avoiding p, or, equivalently, if there is a word with an infinite number of holes which avoids p. If no such word exists, we say the p is k-unavoidable. If p is k-avoidable (resp. k-unavoidable) for all $k \geq 2$, we call it *avoidable* (resp. *unavoidable*).

Examples: $\alpha\beta$ is unavoidable.

$\alpha\alpha$ is unavoidable for partial words.

$\alpha\alpha$ 3-avoidable for fullwords.

cubes $\alpha\alpha\alpha$ are 2-avoidable.

Definition

Definition - *avoidability index, full word avoidability index*

For a given pattern p we define the *avoidability index* $\mu(p)$ as the minimal k such that p is k-avoidable. If p is unavoidable, we say $\mu(p) = \infty$. We call $\mu'(p)$ the *full word avoidability index* of p, defined as the minimal k such that an infinite full word on a k letter alphabet avoids p.

Examples:

$\mu(\alpha\beta) = \infty$

$\mu(\alpha\alpha\beta\beta) = 3$

Every binary pattern p length 7 or greater has $\mu(p) = 2$.

There are patterns p_4 and p_5 such that $\mu'(p_4) = 4$ and $\mu'(p_5) = 5$

Definition

Definition - *DOL (Deterministic O... Lindenmeyer) system*

For a morphism $f : A^* \to A^*$ and $a_0 \in A$ we call the tuple $D = (A, f, a_0)$ a *DOL system* and define the *DOL language* generated by S as the set $\{f^n(a_0) \mid n \in \mathbb{N}\}$

Example:

The Thue-Morse morphism $t(a) = ab$ and $t(b) = ba$ gives the DOL system $(\{a, b\}, t, a)$ generating the language

$$\{\varepsilon, a, ab, abba, abbabaab, abbabaabbaababba, \ldots\}$$

Definition

Definition - *Fixed Point*

For a DOL system (A, f, a_0), we define the *fixed point* as

$$f^{\omega}(a_0) = \lim_{n\to\infty} f^n(a_0)$$

provided the limit exists.

Example:

The Thue-Morse word is the fixed point of the morphism $t(a) = ab$ and $t(b) = ba$.

Definition

Definition - *HDOL system*

For a morphism $g : A^* \to B^*$ with B a secondary alphabet and a DOL system (A, f, a_0), the tuple (A, f, a_0, B, g) is called an *HDOL system*. We define the *HDOL language* generated by H as the set $\{g \circ f^n(a_0) \mid n \in \mathbb{N}\}$

Definition

Definition - *h-injected*

Let A and B be alphabets. For a word $w \in B^+$ and a morphism $h : A^* \to B^*$ we say that w is *h-injected from x* if $x \in A^+$ is a unique word of minimal length such that w is a factor occurring once in $h(x)$ and for all $y \in A^+$ if w is a factor of $h(y)$ then x is a factor of y. We will say w is *h-injected* if such an x exists.

Definition

Definition - *h-postinjected, h-preinjected*

Let A and B be alphabets. For a word $w \in B^+$ and a morphism $h : A^* \to B^*$ we say that w is *h-preinjected from a* (respectively *h-postinjected from a*) if $a \in A$ such that w is compatible with $\text{Pref}(h(a))$ (respectively $\text{Suf}(h(a))$).

Definition

Definition - *h-side-injected*

Let A and B be alphabets. For a word $w \in B^+$ and a morphism $h : A^* \to B^*$ we say that w is *h-side-injected from a* if $a \in A$ such that the number

$$k_a = |\{u \in \text{Pref}(h(a)) | u \uparrow w\}| + |\{u \in \text{Suf}(h(a)) | u \uparrow w\}|$$

is exactly one, and k_b is zero for all other letters $b \in A$.

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