Math 3012-L
Spring 2018
Exam 1
1 Feb
Time Limit: 70 Minutes

Name: Key

This exam contains 8 pages (including this cover page) and 6 questions. There are 0 points in total. Justify all answers. Any expression for a number is acceptable; there is no need to find a decimal representation. Write explanations or proofs clearly and in complete thoughts. Points are reserved for clarity. Use the blank side of paper for scratch work. No calculators or notes may be used.

On my honor, I pledge that I will not give or receive aid in examinations; I will not use unapproved materials in examinations; I will not misrepresent my work or represent the work of another as my own; and I will avoid any activity which will encourage others to violate their own pledge of honor.

Signature: ____________________

Print Name: ____________________

Formal Symbols Crib Sheet

$\neg$	not	$\wedge$	and	$\vee$	or
$\Rightarrow$	implies	$\lightning$	contradiction	$\in$	element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$	equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$	integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv \pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$	complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$	set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$	big-O asymptotic order
2^A	power set of set A	$\lvert A\rvert$	cardinality of set A	A^B	set of functions $B \to A$

The Twelvefold Way:

$|\{f : k \to n\}|$

How many ways to sort k balls into n boxes?

	Arbitrary any sorting	Injective max 1 ball per box	Surjective each box gets ball
Distinct Balls Distinct Boxes	n^k	$\frac{n!}{(n-k)!}$	$n!\left\{ {k \atop n} \right\}$
Identical Balls Distinct Boxes	$\binom{n+k-1}{k}$	$\binom{n}{k}$	$\binom{k-1}{n-1}$
Distinct Balls Identical Boxes	$\sum_{j=0}^{n} \left\{ {k \atop j} \right\}$	1 if $k \leq n$	$\left\{ {k \atop n} \right\}$
Identical Balls Identical Boxes	$p_{\leq n}(k)$	1 if $k \leq n$	$p_n(k)$

1. (a) (3 points) What makes a decision problem **P**? What makes a decision problem **NP**?

A decision problem is P if it can be solved by an algorithm of complexity $\mathcal{O}(n^k)$ for some constant k.

" NP if a certificate can be checked by an algorithm of complexity $\mathcal{O}(n^k)$ for some constant k.

(b) Consider the following decision problem:

Given a list of n positive integers less than $50n$, decide if two distinct numbers in the list multiply to $4n+8$.

Describe an algorithm that can answer the decision problem and estimate the $\mathcal{O}$ complexity of your algorithm. You must state what basic operations you are counting.

Compute $4n+8$.
For every pair x, y in the list:
Compute $x \cdot y$
Check $x \cdot y \stackrel{?}{=} 4n+8$

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This requires 1 multiplication and 1 equality check for every pair. There are $\binom{n}{2}$ pairs.

$\mathcal{O}(n^2)$ multiplications and comparisons

2. (a) (3 points) Circle True or False.

A. For a graph $G = (V, E)$ we have $|E| = \mathcal{O}(|V|^2)$.

TRUE FALSE

B. If S is a set and w is the width of the poset of subsets of S, then $w = \mathcal{O}(|S|^2)$.

TRUE FALSE

The width is $\binom{|S|}{\lfloor |S|/2 \rfloor} \neq \mathcal{O}(|S|^2)$,

C. If $H = (V', E')$ is a subgraph of $G = (V, E)$, then $|E'| = \mathcal{O}(|E|)$.

TRUE FALSE

(b) (3 points) How many subgraphs of the complete graph K_{11} with vertex set $\{0, \ldots, 10\}$ are trees?

11^9 by the Prüfer code count

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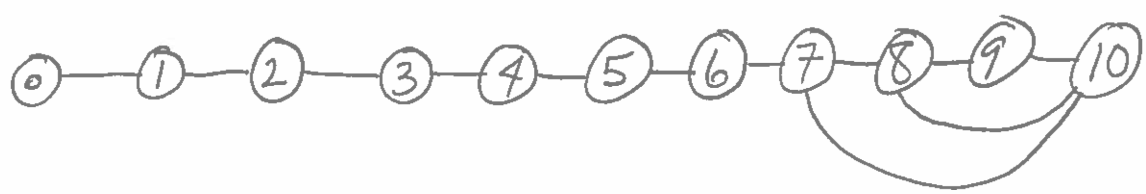
(c) (3 points) Suppose a graph G' has 11 vertices. Recall that the *symmetries* of G' are the graph isomorphisms from G' to itself. What is the maximum number of symmetries G' might have? What is the least number of symmetries G' might have?

G' might have $11!$ symmetries. K_{11} does.

Or it might have only 1 symmetry like this graph:



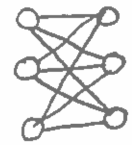
3. (a) (4 points) Circle True or False.

A. If a graph G is planar, then G is also Hamiltonian.
TRUE FALSE

No →

B. If a graph G is 4-colorable, then G is also planar.
TRUE FALSE

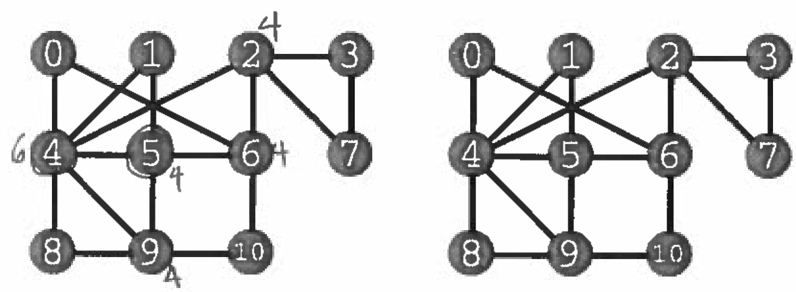
No →



C. Deciding if G is planar is an **NP**-problem.
TRUE FALSE

D. Deciding if G is planar is not a **P**-problem.
TRUE FALSE

Remember $P \subset NP$



(b) (3 points) Consider the graph shown above. (Two copies are provided for your convenience.) Is the graph Eulerian? Justify your claim.

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Yes! The graph is connected and the dgree sequence 64444222222 is all even!

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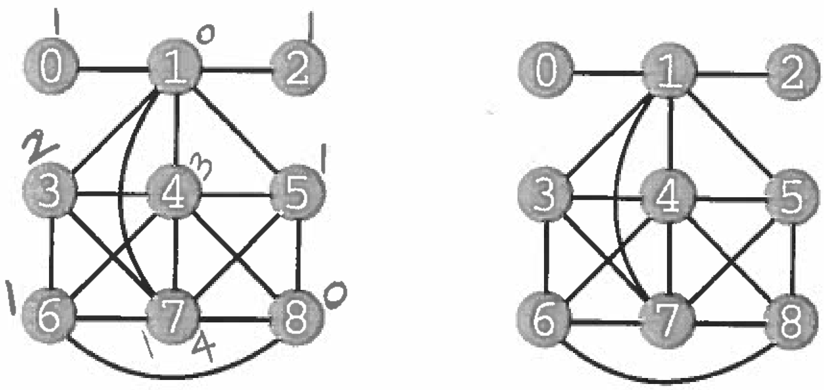
(c) (3 points) Consider the graph shown above. Is the graph Hamiltonian? Justify your claim.

No! Removing 2 would disconnect the graph. If you started at 3, there's no way to get back without using vertex 2 twice.

4. (a) (3 points) What is a k-coloring of a graph? If $G=(V,E)$

A function $f: V \to k$ such that

if u is adjacent to v then $f(u) \neq f(v)$.



(b) (3 points) Consider the graph above. What is the chromatic number of this graph? Explain.

$\boxed{5.}$

There is a 5-coloring: Color 1, 8 as 0
0, 2, 5, 6 as 1
3 as 2
4 as 3
7 as 4

And you need 5 colors since; 1, 5, 8, 6, 3 is a 5-cycle it requires 3 colors.
Vertex 4 is adjacent to all of these so requirss a 4th color.
Vertex 7 is adjacent to all of those, so it requires a 5th color.

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(c) BONUS: Suppose G is known to have chromatic number 3 and has vertex set $\{0,\ldots,9\}$. Both $\{0,1\}$ and $\{1,2\}$ are edges in G, but the other edges of G are not known. How many possible 3-colorings of G are consistent with this information, up to relabeling of the colors?

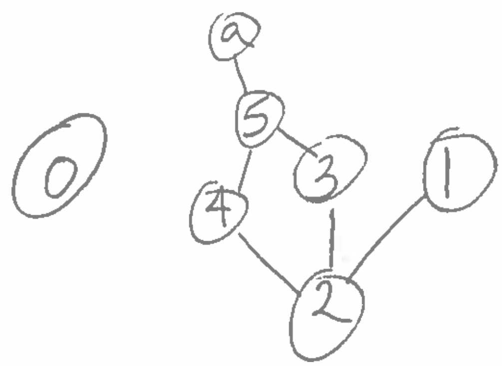
3pts

$$\left\{ {10 \atop 3} \right\} - 2 \cdot \left\{ {9 \atop 3} \right\} + \left\{ {8 \atop 3} \right\}$$

Include exclude the partitions of 10 based on if 0&1 are together and if 1&2 are together.

5. (a) (3 points) Draw the Hasse diagram for the poset

$$\{(a,a),(0,0),(1,1),(3,3),(4,4),(5,5),(2,1),(2,3),$$
$$(2,4),(2,5),(2,a),(3,a),(3,5),(4,a),(4,5),(5,a)\}$$



(b) (3 points) A graph has degree sequence $(4,4,4,4,3,2,2,2,1,1,1)$. Must it be planar, must it be nonplanar, or might it be either? Explain.

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It must be planar by Kuratowski's Theorem.
There are only 4 vertices with degree ≥ 4, so no K_5 homeorphic subgraph can exist.
There are only 5 vertices with degree ≥ 3 so no $K_{3,3}$ homeomorphic subgraph can exist.

Use 12-fold way!

6. The 2018 Winter Olympics were held in PyeongChang, Soth Korean.

(a) (2 points) Competing were 2,922 athletes representing exactly 92 National Olympic Committees. How many ways might the 2,922 different athletes have come from the 92 different National Olympic Committees if we track which athlete competes for which nation?

2922 distinct balls → 92 distinct boxes

surject

$$92! \left\{ {2922 \atop 92} \right\}$$

(b) (3 points) Athletes competed in 102 events in 15 sports, with a gold, silver, and bronze medal awarded in each event. How many ways might the medals have been awarded to the 92 National Olympic Committees if we track the number of each type of medal?

Gold medals: 102 identical balls → 92 distinct boxes

in $\binom{102+92-1}{102}$ ways.

Silver & Bronze are similar: $\binom{102+92-1}{102}^3$

(c) (3 points) In fact Norway had the highest total medal count with 39, and only 30 National Olympic Committees won any medals. How many ways may the remaining 267 medals have been distributed among the other 29 nations? Note no nation but Norway won more than 38.

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Solutions to $x_1 + \dots + x_{29} = 267$ with $1 \le x_i \le 38$.

Include exclude the surjective solutions with some $x_i \geq 38$

$+5$

$$\sum_{k=0} (-1)^k \binom{29}{k} \cdot \binom{267-1-38k}{29-1}$$

choose k countries with ≥ 38

ways to surject $267-38k$ identical balls onto 29 boxes.

(d) BONUS: What nation had the second highest medal count?

1pt

Germany