Math 3012-L
Spring 2018
Exam 3
5 April
Time Limit: 70 Minutes

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This exam contains 7 pages (including this cover page) and 5 questions. There are 33 points in total. Justify all answers. Any computable expression for a number is acceptable; there is no need to find a decimal representation. Write explanations or proofs clearly and in complete thoughts. Points are reserved for clarity. Use the blank side of paper for scratch work. No calculators or notes may be used.

On my honor, I pledge that I will not give or receive aid in examinations; I will not use unapproved materials in examinations; I will not misrepresent my work or represent the work of another as my own; and I will avoid any activity which will encourage others to violate their own pledge of honor.

Signature: ____________________

Print Name: ____________________

Formal Symbols Crib Sheet

$\neg$	not	$\wedge$	and	$\vee$	or
$\Rightarrow$	implies	$\lightning$	contradiction	$\in$	element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$	equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$	integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv \pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$	complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$	set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$	big-O asymptotic order
2^A	power set of set A	$\lvert A\rvert$	cardinality of set A	A^B	set of functions $B \to A$

The Twelvefold Way:

$|\{f : k \to n\}|$

How many ways to sort k balls into n boxes?

	Arbitrary any sorting	Injective max 1 ball per box	Surjective each box gets ball
Distinct Balls Distinct Boxes	n^k	$\frac{n!}{(n-k)!}$	$n!\left\{{k \atop n}\right\}$
Identical Balls Distinct Boxes	$\binom{n+k-1}{k}$	$\binom{n}{k}$	$\binom{k-1}{n-1}$
Distinct Balls Identical Boxes	$\sum_{j=0}^{n} \left\{{k \atop j}\right\}$	1 if $k \leq n$	$\left\{{k \atop n}\right\}$
Identical Balls Identical Boxes	$p_{\leq n}(k)$	1 if $k \leq n$	$p_n(k)$

1. (9 points) Give a closed formula for generating function for the sequences with the n^{th} term described below. Remember the geometric series:

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

(a) The number of strings on $\{1, 2, 3\}$ with length n.

There are 3^n strings length n so

$$\sum_n 3^n x^n = \sum (3x)^n = \frac{1}{1-3x}$$

(b) The number of strings on $\{1, 2, 3\}$ with length n where the digits are non-increasing.

Pick the # of 3s Pick the # 2s Pick the # of ones

$$\frac{1}{1-x} \cdot \frac{1}{1-x} \cdot \frac{1}{1-x}$$

(c) The number of strings on $\{1, 2, 3\}$ whose digits sum to n.

A single digits adds 1, 2, or 3 to the sum.

$D = x + x^2 + x^3$ for each digit.

There are any # of digits allowed and these are disjoint cases:

$$\sum_n D^n = \frac{1}{1-D} = \frac{1}{1-(x+x^2+x^3)}$$

2. (6 points) (a) What is a group of permutations?

A nonempty set of permutations of some set X such that contains all the compositions and inverses of its elements.

(b) If $\tau = (153)(298)$ is a permutation, what is $\tau(3)$?

$\tau(3) = 1$

(c) Compute the composition of the permutations.

$$(1234)(345)(234) = ?$$

$(12)(354)$

3. (6 points) Find a closed formula in n for sequence a_n with $a_0 = 0$, $a_1 = 2$, and which satisfies the the recurrence

$$a_{n+2} = 6a_{n+1} - 8a_n.$$

the characteristic polynomial is

$$\lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

So any possible solution has the form

$$a_n = c_0 2^n + c_1 4^n$$

Plugging in $n=0$ and $n=1$ we solve for c_0 and c_1

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\text{rref} \begin{vmatrix} 1 & 1 & 0 \\ 2 & 4 & 2 \end{vmatrix} = \text{rref} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \end{vmatrix} = \cancel{\text{rref}} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

So $\boxed{a_n = 4^n - 2^n}$

4. (6 points) It is known that every integer larger than 1500 can be written as a *non-negative* integer linear combination of 119, 42, and 66, i.e.

$$(*) \quad 119x + 42y + 66z = n$$

with integers $x, y, z \geq 0$ has a solution for every $n \geq 1500$. But not every n has a solution. Explain how a generating function could be used to find all the numbers that **cannot** be written as a non-negative integer linear combination of 119, 42, and 66.

The generating function for the number of solutions to Equation $(*)$ is

$$f(x) = \frac{1}{1-x^{119}} \cdot \frac{1}{1-x^{42}} \cdot \frac{1}{1-x^{66}}$$

Pick #119s
then # of 42s
then # of 66s.

so the coefficient of x^n is the number of solutions.

If there are no solutions, then the coefficient is 0. So expand $f(x)$ to ~~[illegible]~~ x^{1500} and find all powers of x with a coefficient of 0.

5. (6 points) The symmetry group of the fish graph has 4 symmetries: you can flip the tail, the head, neither, or both.

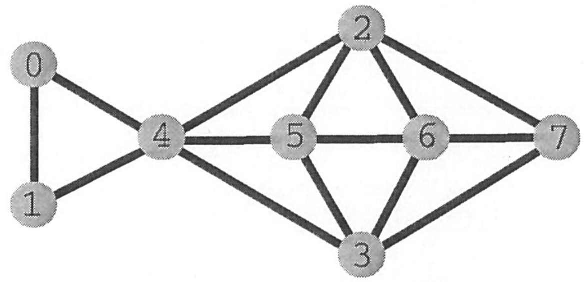
Permutation Group

id

(01) (2)(3)(4)(5)(6)(7)

(0)(1)(23) (4) ... (7)

(01)(23)(4)...(7)



(a) How many distinct ways could you paint the 8 vertices of the fish graph blue, green, cyan, turquoise, or navy, up to symmetries of the fish?

Using Burnside Lemma we average the fixed colorings over the group:

$$\frac{5^8+5^7+5^7+5^6}{4}$$

(b) What is the Polya cycle index of the permutation group of fish graph symmetries?

$$\frac{x_1^8 + 2x_2x_1^6 + x_2^2x_1^4}{4}$$