Math 3012L **Name:** ____________

Spring 2018
Final
26 April 2018
Time Limit: 14:50 to 17:40

This exam contains 9 pages (including this cover page) and 8 questions. There are 48 points in total. Justify all answers. Any computable expression for a number is acceptable; there is no need to find a decimal representation. Write explanations and proofs clearly and in complete thoughts. Points are reserved for clarity. Use the blank side of paper for scratch work. No calculators or notes may be used.

On my honor, I pledge that I will not give or receive aid in examinations; I will not use unapproved materials in examinations; I will not misrepresent my work or represent the work of another as my own; and I will avoid any activity which will encourage others to violate their own pledge of honor.

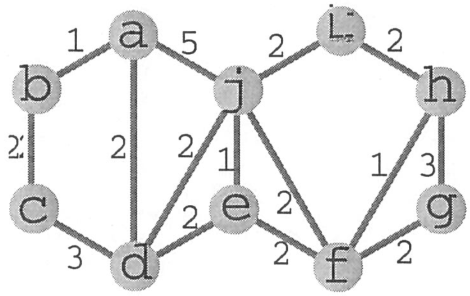
Signature: ____________

Print Name: ____________

Formal Symbols Crib Sheet

$\neg$	not	$\wedge$	and	$\vee$	or
$\Rightarrow$	implies	$\lightning$	contradiction	$\in$	element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$	equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$	integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv \pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$	complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$	set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$	big-O asymptotic order
2^A	power set of set A	$\lvert A\rvert$	cardinality of set A	A^B	set of functions $B \to A$

1. (6 points) Consider the weighted graph shown below. There is another copy on the 12fold way page for your convenience.



   (a) Is the graph Hamiltonian? Justify your answer.

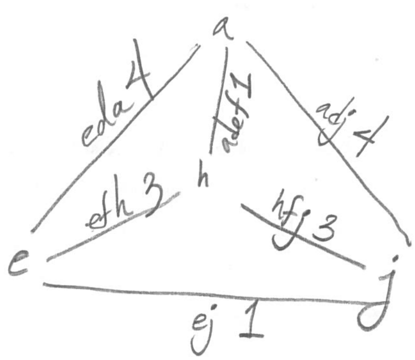
   Yes. abcdefghija is a Hamiltonian cycle.

   (b) Find a minimal weight closed walk that contains every edge of the graph and its weight. (The total weight of all the edges in the graph is 32.)

   The odd degree vertices a, e, h, j prevent an Eulerian circuit. Add copies of paths between these vertices in minimal weight pairs to produce an Eulerian multigraph.

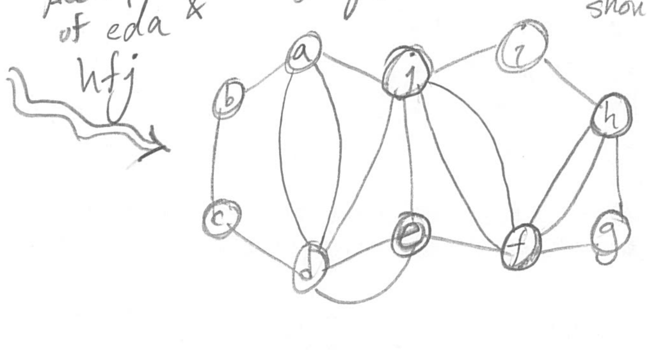
   There are two weight 7 minimal weight perfect matchings of the odd degree vertices.

   Any minimal weight closed walk is weight 32 + 7 = 39.



   Any Eulerian circuit of this multigraph should do.

   Add copies of eda & hfj



   E.g. abcdadedjefjfhfg-hija

2. (6 points) Choose one of the following statements and prove it. Circle the statement you are proving.

   (a) If the edges of the complete graph K_6 are colored red and blue, then no matter how you color there must be a triangle with either all red or all blue edges.

   (b) The complete graph K_5 is not planar.

   (c) The planar diagram of any connected planar graph splits the plane into $R = E - V + 2$ regions.

(a) Label a vertex 0. Since $\deg(0) = 5$ it has at least $\lceil \frac{5}{2} \rceil = 3$ incident edges of the same color. Say red. Let 1, 2, 3 be the labels of the neighbors of 0 along these red edges. If the edges 12, 23, 13 are all blue, there is a blue triangle. If any of them is red, it must form a red triangle with vertex 0.

(b) A connected planar graph has a diagram with V vertices, E edges, and R regions, and by Euler $V-E+R=2$. If there are at least 3 edges then every region has at least 3 edges bounding it. Then

$$2E = \sum_{r \text{ a Region}} (\text{\# bounding edges of } r) \geq 3R.$$

But $V-E+R=2$ and $2E \geq 3R$ imply $E \leq 3V-6$.
Assume to the contrary that K_5 had a planar diagram. Then it has 5 vertices and $10 = \binom{5}{2}$ edges, so we would have $10 \leq 3 \cdot 5 - 6 = 9$, a contradiction. So K_5 must not be planar.

(c) Induct on the number of edges of a graph. As a base case note that the graph with a single vertex and no edges has 1 region for the plane and $1 - 0 + 1 = 2$.
Assume that $V-E+R=2$ for any connected planar graph diagram with $E=k$ edges.
Let G be a planar graph diagram with k+1 edges. Let e be an edge incident to a leaf (a degree 1 vertex) if G is a tree, or else any edge whose removal does not disconnect G if G is not a tree. Let G' be the graph made by removing e from G (along with its leaf if applicable). Then G' is planar, connected with k edges so by inductive hypothesis it has $V-k+R=2$.
So then G has $(V+1)-(k+1)+R=2$ if we removed a leaf, or e split a region in two and G has $V-(k+1)+(R+1)=2$.

3. (6 points) The (caseless) English alphabet $\{A, B, \ldots, Z\}$ consists of 26 different letters with 5 vowels $\{A, E, I, O, U\}$ and 21 consonants. You may wish to consult the 12-fold table.

(a) How many sets of 8 letters are there?

There are $\binom{26}{8}$ size 8 subsets of a 26 size set.

(b) How many ways can the English alphabet be partitioned into exactly 5 parts?

The Stirling number $\left\{ {26 \atop 5} \right\}$ counts sortings of 26 distinct letter-labeled balls surjectively into 5 indistinct boxes.

(c) How many ways can the English alphabet be partitioned into exactly 5 parts so that the vowels are each in a different part of the partition?

For each consonant, choose a vowel.

$$5^{21}$$

(d) How many collections of 52 (possibly repeated) letters have exactly 10 vowels?

There are 42 consonants from 21 and 10 vowels from 5 options.

There are $\binom{42+21-1}{42}$ collections of 42 consonants via the kids-and-candy count.

Then there are $\binom{10+5-1}{10}$ collections of 10 vowels.

$$\binom{62}{42} \cdot \binom{14}{10}$$

4. (6 points) (a) Explain the **P** vs **NP** question. Give an example of a decision problem known to be **P**, and an example of a decision problem known to be **NP** but not known to be **P**.

A P decision problem has a $\Theta(n^k)$ algorithm for constant k

A NP decision problem has a certification process with a $\Theta(n^k)$ algorithm for constant k.

$P \subseteq NP$, but it is unclear if $NP = P$ or $NP \neq P$.

Determiny in a graph is Eulerian is P, but determiny if a graph is Hamiltonian is NP, but not known to be P.

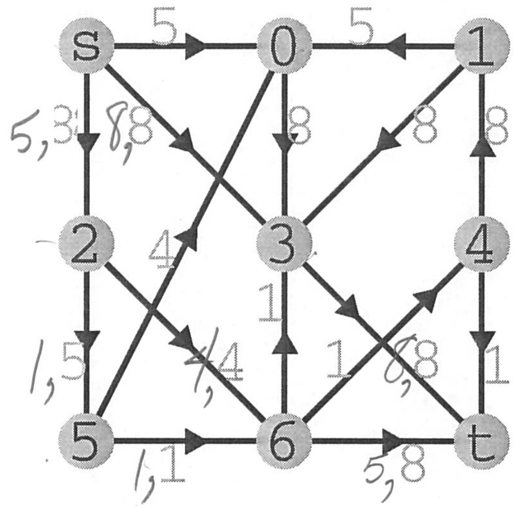
(b) The set of all subsets of $\{A, C, G, T\}$ is called the power set. The power set of $\{A, C, G, T\}$ has the partial order $\subset$ with $x \subset y$ if x is a subset of y. What is the fewest number of chains needed to partition the power set of $\{A, C, G, T\}$?

$\binom{4}{\lceil 4/2 \rceil} = 6$ is the width of the poset of subsets of a set, so by Dilworth's Theorem 6 chains, but no fewer can partition all the subsets of $\{A, C, G, T\}$.

5. (6 points) Consider the network shown below. There is another copy on the 12fold way page for your convenience. The arcs and their capacities are also summarized in the table.

Arc	s0	s2	s3	0,3	1,0	1,3	2,5	2,6	3t	4,1	4,t	5,0	5,6	6,3	6,4	6t
Capacity	5	8	8	8	5	8	5	4	8	8	1	4	1	1	1	8
Flow		5	8				1	4	8				1			5

Find a maximum flow and a corresponding minimal cut for the network. Give the volume of the flow and the capacity of the cut.



There is a volume 13 flow
And a capacity 13 = 1+4+8 cut
can be given by
{s0235} {146t}

6. (6 points) (a) Give a closed form for the generating function of the sequence of alternating ones and twos: $1, 2, 1, 2, 1, 2, \ldots$

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \ldots$$

$$= (1 + x + x^2 + x^3 + \ldots) + (x + x^3 + x^5 + \ldots)$$

$$= \frac{1}{1-x} + \frac{x}{1-x^2}$$

(b) What is the number of integer partitions of 256 into even sized parts? (Hint: use a generating function.)

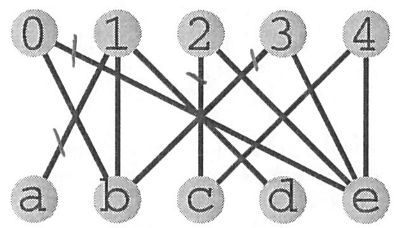
You could divide by 2 to biject to $p(128)$!

coefficient of x^{128} in $\prod_{k \geq 1} \frac{1}{1-x^k}$

OR equivalently

coefficient of x^{256} in $\prod_{k \geq 1} \frac{1}{1-x^{2k}}$

7. (6 points) Consider the graph shown below. There is another copy on the 12fold way page for your convenience.



(a) Compute a maximal matching for the graph.

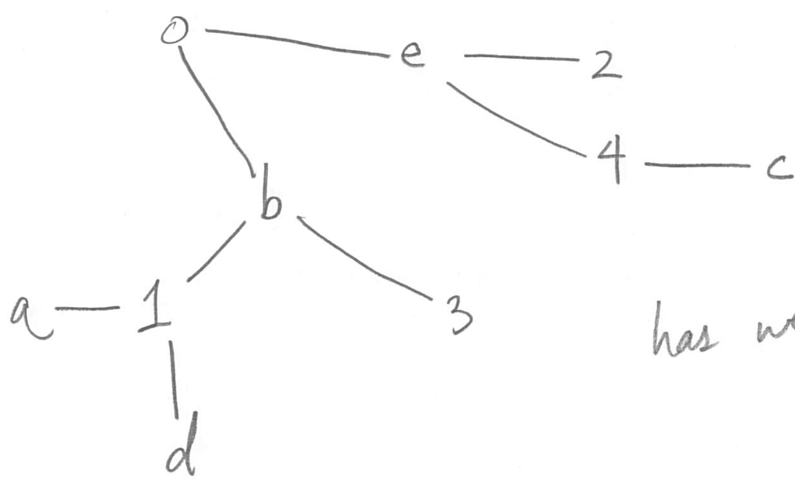
At most 4 can be matched.

0b 1a 2c 3b is such a matching.

(b) Suppose the edges have weights as described in the following table. Compute a minimal spanning tree and its weight.

Edge	0b	0e	3b	3e	1b	2e	4e	4c	2c	1d	1a
Weight	1	2	3	4	5	6	7	8	9	10	11
	✓	✓	✓	×	✓	✓	✓	✓	×	✓	✓

Using Kruskal.



has weight $\binom{12}{2}-4-9$

$= 66-13=53$

8. (6 points) (a) How many ways could you rearrange the letters in the phrase

CAT SCRATCH STITCHES

into a string that has neither RAT nor TRASH as a substring?

Letter	Count
A	2
C	4
E	1
H	2
I	1
R	1
S	3
T	4
Σ	18

Treat all letter positions as initially distinct, then divide out the symmetries of swapping identical letters.

- There are $18!$ permutations of the letters.
- There are $1\cdot2\cdot4$ ways to pick the letters for RAT and then $16!$ permutations with RAT and the remaining 15 letters.
- Similarly $4\cdot1\cdot2\cdot3\cdot2\cdot14!$ ways to pick letters for TRASH and permute it with the remaining letters.
- Since there is only 1 R, you can't have both simultaneously.

Distinct Anagrams: $\dfrac{18! - 1\cdot2\cdot4\cdot16! - 4\cdot2\cdot3\cdot2\cdot14!}{4!\,4!\,3!\,2!\,2!}$

(b) How many length 6 strings of $\{R, O, C, K, S\}$ are distinct up to cyclic permutation? E.g. ROC and OCR and CRO are length 3 strings which are the same up to cyclic permutation.

perm	index
(0)(1)(2)(3)(4)(5)	x_1^6
(012345)	x_6
(024)(135)	x_3^2
(03)(14)(25)	x_2^3
(042)(153)	x_3^2
(054321)	x_6

The group of 6 rotations acts on the letter positions with Polya Cycle index: $\dfrac{x_1^6 + x_2^3 + 2x_3^2 + 2x_6}{6}$

so by Burnside's Lemma there are

$$\frac{5^6 + 5^3 + 2\cdot5^2 + 2\cdot5}{6} = 2635$$

distinct length 6 strings using 5 letters up to cyclic permutation.